Dec 3, 2018

Applications of MVT:

MVT :

Let $f: [a,b] \longrightarrow \mathbb{R}$, $a < b$

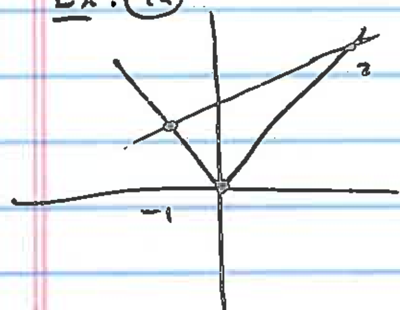
f be continuous on $[a,b]$

f be diff'ble on (a,b)

Then $\exists\, c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Ex. (1a)



$f(x) = |x|$

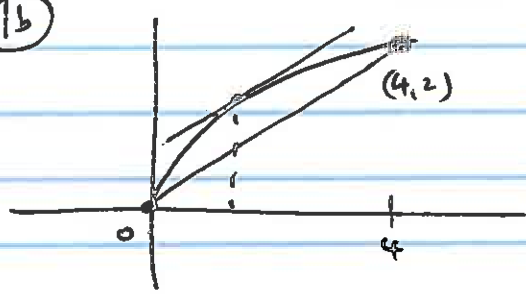
$$\frac{f(2)-f(-1)}{2-(-1)} = \frac{2-1}{3} = \frac{1}{3}$$

$\exists$ no c s.t. $f'(c) = \frac{1}{3}$

since f is not diff'ble on all of $(-1,2)$.

MVT doesn't apply.

(1b)



$g(x) = \sqrt{x}$ MVT applies even though g is not diff'ble at 0

$$g'(x) = \frac{1}{2\sqrt{x}} \text{ if } x > 0$$

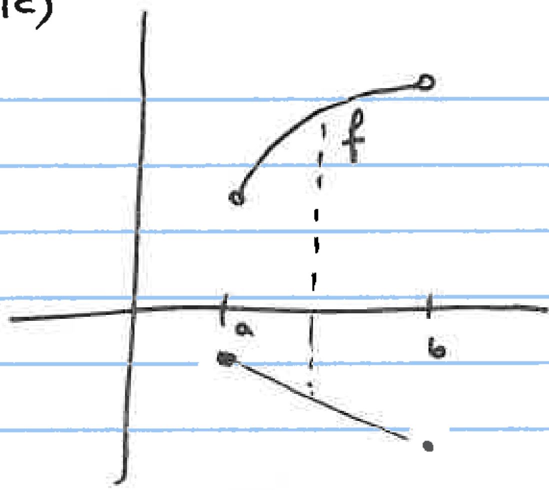
Find c s.t.

$$g'(c) = \frac{g(4)-g(0)}{4-0}$$

$$\frac{1}{2\sqrt{c}} = \frac{2-0}{4-0} = \frac{1}{2}$$

$$\sqrt{c} = 1 \implies c = 1.$$

Ex. 1(c)



f diff'ble on (a,b)

f not continuous at a, b

$\nexists$ no c s.t.

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

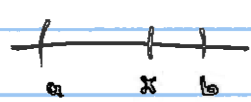
MVT doesn't apply

Corollaries of MVT.

Thm 6.2.6 ① $\left.\begin{array}{l} f:[a,b] \to \mathbb{R} \text{ continuous} \\ f'(x) \equiv 0 \end{array}\right\} \Longrightarrow f(x) \equiv \text{constant} = f(a)$

Thm 6.2.6 ② $\left.\begin{array}{l} \left.\begin{array}{l} f:[a,b] \to \mathbb{R} \\ g:[a,b] \to \mathbb{R} \end{array}\right\} \text{continuous} \\ f, g \text{ diff'ble on } (a,b) \\ f'(x) = g'(x) \end{array}\right\} \Longrightarrow f(x) = g(x) + C$ for some $C \in \mathbb{R}$.

Proof: ①



$$\frac{f(x) - f(a)}{x - a} = f'(c) \quad \text{for some } c \in (a,x)$$

$$= 0$$

$$f(x) - f(a) = 0 \quad \forall x \in [a,b].$$

$$f(x) \equiv f(a) \quad \forall x \in [a,b].$$

② Let $h(x) = f(x) - g(x)$

$$h'(x) = f'(x) - g'(x) = 0$$

$$f(x) - g(x) = h(x) = \text{constant} = C.$$

Thm 6.2.8, Exc #8, #9

Defn A function $f: I \to \mathbb{R}$ is
- called strictly increasing if $x > y \Rightarrow f(x) > f(y)$;
- called increasing if $x \geq y \Rightarrow f(x) \geq f(y)$.

Similarly for decreasing, strictly decreasing.

Prop Let $f: [a,b] \to \mathbb{R}$, be diff'ble on $[a,b]$

a) $f'(x) > 0$ on $[a,b] \Rightarrow f$ is strictly increasing on $[a,b]$

Exc 8
b) $f'(x) \geq 0$ on $[a,b] \Rightarrow f$ is increasing on $[a,b]$

c) $f'(x) \geq 0$ on $[a,b] \Leftarrow f$ is increasing on $[a,b]$.

(Similarly for f' negative, f decreasing)

Ex $f(x) = x^3$ counter example for "f strictly increasing $\Rightarrow f'(x) > 0$"

Exc 9

$\forall a, b \in \mathbb{R}$ $a < b \Rightarrow a^3 < b^3$

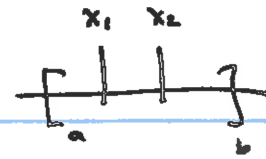
but $f'(0) = 3 \cdot 0^2 = 0$.

Hence

f strictly increasing $\not\Rightarrow f'(x) > 0 \ \forall x$

Proofs

a) & b)



Let $a \leq x_1 < x_2 \leq b$.

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} = f'(c) \quad \text{for some } c \in (x_1, x_2)$$

a) $$\frac{f(x_1)-f(x_2)}{x_1-x_2} = f'(c) > 0$$

$$x_1 < x_2$$

$$x_1 - x_2 < 0$$

$$\Rightarrow f(x_1) - f(x_2) < 0$$

$$f(x_1) < f(x_2)$$

b) In the above proof, all "<" replaced by "≤"
(except $x_1 < x_2$, $x_1 - x_2 < 0$)

c) WTS f increasing $\Rightarrow f'(x) \geq 0$

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \quad \text{for any } c \in [a,b].$$

$$f \uparrow,\ x \geq c \Rightarrow f(x) \geq f(c)$$

$$\Rightarrow \frac{f(x)-f(c)}{x-c} \geq 0 \quad (x \neq c)$$

$$x \leq c \Rightarrow f(x) \leq f(c)$$

$$\frac{f(x)-f(c)}{x-c} \geq 0 \quad (x \neq c)$$

Cor 4.2.5
Thm 5.1.8 $$\lim \frac{f(x)-f(c)}{x-c} \geq 0 \Rightarrow f'(c) \geq 0 \text{ for all } c \in [a,b].$$

This proof doesnot carry out with ">":

$$\frac{f(x)-f(c)}{x-c} > 0 \quad \not\Rightarrow \quad \underbrace{\lim_{x\to c} \frac{f(x)-f(c)}{x-c} > 0}$$

$$\Downarrow$$

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \geq 0$$

$$\left(\text{Cf.} \quad \frac{1}{n} > 0, \quad \lim \frac{1}{n} = 0. \right)$$

Exercises from the book.

Exc 5 b)

Prove that $\forall x > 1$ $\quad \overbrace{\frac{x-1}{x} < \ln x}^{\text{I do}} \overbrace{< x-1}^{\text{HW}}.$

Let $f(x) = \ln x - \frac{x-1}{x} = \ln x - 1 + \frac{1}{x}$ for $x \in [1,\infty)$

$$f'(x) = \frac{1}{x} - 0 - \frac{1}{x^2} = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$$

$$x > 1 \implies \frac{x-1}{x^2} > 0$$

$f'(x) > 0$

$f(x)$ is strictly increasing

$$f(1) = \ln 1 - \frac{1-1}{1} = 0.$$

if $x > 1$ $\quad f(x) > f(1) = 0 \qquad \ln x - \frac{x-1}{x} > 0$

$$\ln x > \frac{x-1}{x}$$

Exc 5g) Prove that $|\cos x - \cos y| \leq |x-y|$.

$f(x) = \cos x$, $f'(x) = -\sin x$.

if $x \neq y$ $\exists\, c \in$ between x & y s.t.

$$f'(c) = \frac{f(x)-f(y)}{x-y} \qquad (MVT).$$

$$-\sin c = \frac{\cos x - \cos y}{x-y}$$

$$1 \geq |-\sin c| = \left|\frac{\cos x - \cos y}{x-y}\right|$$

$$|x-y| \geq |\cos x - \cos y| . \text{ for } x \neq y.$$

5j) Prove $\left|\frac{\sin ax - \sin bx}{x}\right| \leq |a-b|$, if $x \neq 0$

Fix $x \neq 0$. $a < b$.

Let $f(u) = \sin(ux) : [a,b] \longrightarrow \mathbb{R}$

$$f'(u) = x\cos(ux)$$

$$x\cos(cx) = f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\sin bx - \sin ax}{b-a} \quad \text{by MVT.}$$

$$|x| \geq |x\cos cx| = \left|\frac{\sin bx - \sin ax}{b-a}\right|$$

$$|b-a| \geq \left|\frac{\sin bx - \sin ax}{x}\right|$$

p256 Exc #10 f diff'ble on $(0,1)$

continuous on $[0,1]$.

$f(0) = 0$.

$f' \uparrow$.

Let $g(x) = \frac{f(x)}{x}$

Prove. $\Longrightarrow g \uparrow$.

<u>Sol'n</u>

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \leq f'(x)$$ (with $f(0) \to 0$)

for some $c \in (0, x)$

$0 < c < x$

$$\frac{f(x)}{x} \leq f'(x)$$

$$f \leq f' \cdot x$$

$$0 \leq f' \cdot x - f$$

$$0 \leq \left(\frac{f}{x}\right)' = \frac{f' \cdot x - f}{x^2}.$$

$$\frac{f}{x} = g \uparrow$$