Nov 30, 2018

To Finish 6.1.

p246 **Exc 7d** $f(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

if $x > 0$ $f(x) = x^2$, $f'(x) = 2x$

if $x < 0$ $f(x) = -x^2$, $f'(x) = -2x$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x|x| - 0}{x - 0} = \lim_{x \to 0} |x| = 0 = f'(0)$$

$f'(x) = 2|x|$ defined on all of $\mathbb{R}$.

p247 **Exc #13** $f, g, h : I \to \mathbb{R}$ diff'ble. To show fgh is diff'ble.

Sol^n: proved Th 6.1.7 $f \cdot g$ diff'ble on I and

$(f \cdot g)' = f'g + f \cdot g'$

(fg) diff'ble & h diff'ble $\overset{\text{Th. 6.1.7}}{\Longrightarrow}$ $(fg) \cdot h$ diff'ble.

$$((fg) \cdot h)' = (fg)' \cdot h + (fg) h'$$
$$= (f'g + fg')h + (fg)h'$$
$$= f'gh + fg'h + fgh'.$$

Exc 14 $(f \circ g \circ h)' = \frac{d}{dx}(f(g(h(x))))$

$$= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Proof of diff'blity is similar to Exc #13, by using the chain rule Thm 6.1.10

6.2 First Derivative test 6.2.1

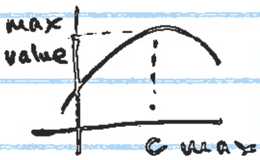
Rolle's Thm 6.2.2

Mean Value Thm. 6.2.3

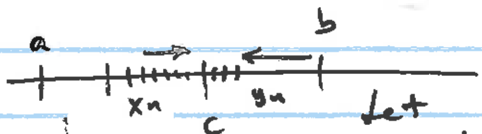
Applications of MVT.

(1st D. Test) **Thm 6.2.1** Let $f:(a,b)\to\mathbb{R}$ be diff'ble on (a,b).
If $c\in(a,b)$ is either a maximum or minimum of f over (a,b).
Then $f'(c)=0$.

Proof **Case 1** $f(c)\geq f(x)$ $\forall x\in(a,b)$ c is max.



$a<c<b$



Let $c-\frac{1}{n}=x_n$ when $n>\frac{1}{c-a}$; Let $y_n=c+\frac{1}{n}$ when $n>\frac{1}{b-c}$

Since $f'(c)$ exists $f'(c)=\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}$

Thm 6.1.3 : $\frac{f(x_n)-f(c)}{x_n-c}\longrightarrow f'(c)$ as $n\to\infty$

$f(x_n)\leq f(c)$ $f(x_n)-f(c)\leq 0$

$x_n-c=-\frac{1}{n}<0$

$0\leq\frac{f(x_n)-f(c)}{x_n-c}\xrightarrow[\lim n\to\infty]{} f'(c)\geq 0.$ Cor 4.2.5

Thm 6.1.3 $\quad \dfrac{f(y_n)-f(c)}{y_n \to c} \longrightarrow f'(c)$ as $n\to\infty$

$f(y_n) \le \underbrace{f(c)}_{\text{max value}} \qquad f(y_n)-f(c)\le 0$

$\qquad\qquad y_n - c = \frac{1}{n} > 0$

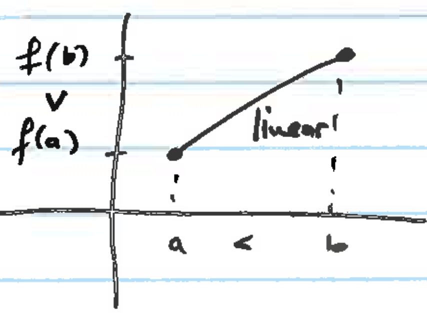
$$0 \geq \frac{f(y_n)-f(c)}{y_n - c} \xrightarrow[\lim n\to\infty]{} f'(c) \le 0$$

Cor. 4.2.5.

$\Longrightarrow f'(c)=0$ #

Case 2 $\quad f(x)\geq f(c)\ \forall x\in(a,b)$, c min. Use $g(x)=-f(x)$, Use Case 1: $g'(c)=0=-f'(c)$

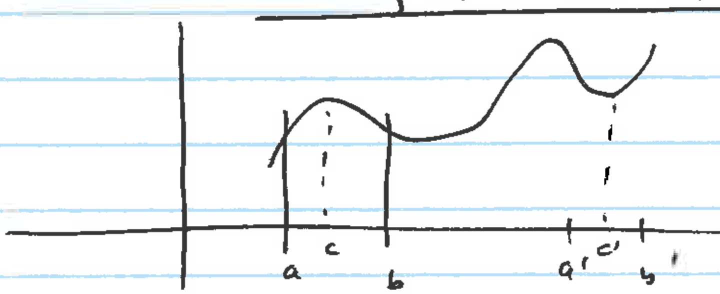
Remarks: ① It doesn't work at the end pts



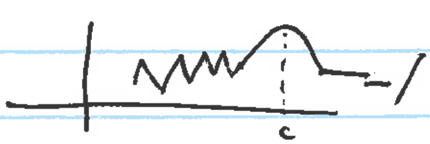
f has max at $b=c$.

$f'(b)\neq 0$

② Local max/min is sufficient as long as in the interior



③ You only need diff'blity of f at c



Ex find Max/min values of

$$f(x) = x^3 - 3x^2 - 9x + 4 \quad \text{on } [0,4].$$

·) f must attain its max & min values
[0,4] closed & bdd, f cont.
Ext. V. Th $\Rightarrow$ f must attain its max & min.

·) f diffble on $(0,4)$ (polynomial function)
Consequence of Thm 6.1.7

$f'(x) = 3x^2 - 6x - 9$

$0 = f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3)$
$= 3(x-3)(x+1)$

$f'(x) = 0 \iff x = 3, -1$

$3 \in (0,4)$
$-1 \notin (0,4)$

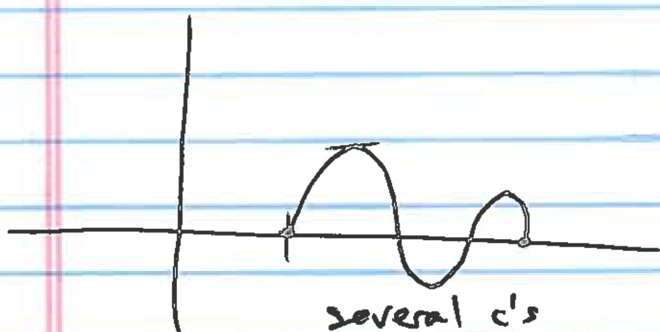
	f
0	4
4	−16
3	−23

max value 4 at 0
min value −23 at 3

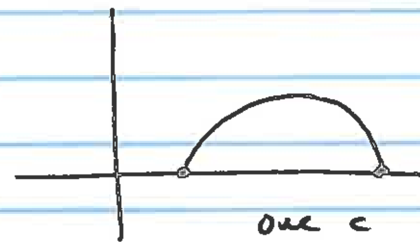
<u>Rolle's Thm</u> Let $a<b$.

If $f:[a,b] \longrightarrow \mathbb{R}$ is continuous and f is diff'ble on (a,b), & $f(a)=f(b)=0$,

then $\exists\ c \in (a,b)$ s.t. $f'(c)=0$.



several c's



one c

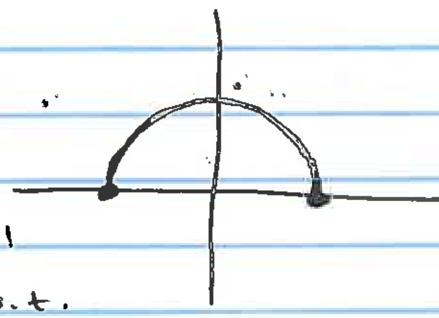
Rolle's Thm applies: $f(x) = \sqrt{1-x^2}$.

$f'(x) = \frac{-x}{\sqrt{1-x^2}}$

f not diff'ble at ± 1

$\exists$ only one $c=0$ s.t.

$f'(c)=0$.



<u>Proof of Rolle's Thm</u>

Since $f:[a,b] \to \mathbb{R}$ continuous, $[a,b]$ is compact, f must attain its max & min values:

$\exists\ x_1, x_2 \in [a,b]$, $\forall x \in [a,b]$

$$f(x_1) \le f(x) \le f(x_2).$$

<u>Case 1</u> either x_1 or $x_2 \in (a,b) = \text{int}[a,b]$. then

you take $c = x_1$ or x_2 which ever one in the interior. Say $x_i \in (a,b)$.

$c = x_i \in (a,b)$

$f'(c) = f'(x_i) \equiv 0$ by first derivative test.

<u>Case 2</u> Both x_1 and x_2 are at the boundary

$\{x_1, x_2\} \subseteq \{a, b\}$.

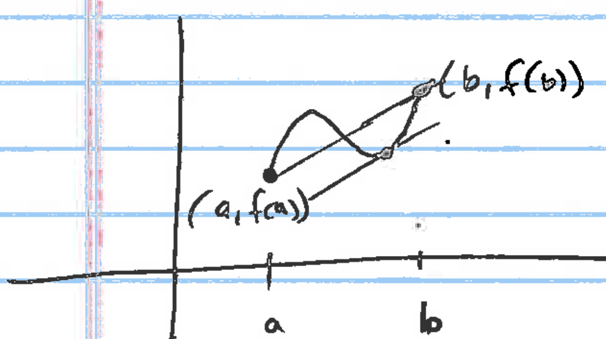
$f(a) = f(b) = 0$

$$0 = f(x_1) \leq f(x) \leq f(x_2) = 0$$

$f \equiv 0$ on all of $[a,b]$.

For all $c \in (a,b)$, $f'(c) \equiv 0$. #

6.2.3 <u>Thm:</u> Mean Value Thm



Let $a < b$,
$f : [a,b] \to \mathbb{R}$ be continuous,
and f be diffble on (a,b).
Then $\exists\, c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

<u>Proof of</u> MVT follows Rolle's Thm.

$$\text{Let } g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

$g(x)$ is continuous on $[a,b]$ by Thms from Chap 5

$g(x)$ is diffble on (a,b) " " " 6.1

$$g(a) = f(a) - \frac{f(b)-f(a)}{b-a}\underbrace{(a-a)}_{0} - f(a) = 0.$$

$$g(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) - f(a) = 0.$$

By Rolle's Thm $\exists\ c \in (a,b)$ s.t. $g'(c) = 0$

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \cdot 1 - 0$$

$$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad \#.$$

Many Applications. on Monday.