6.1 Continue

Remark: We define differentiability of a function $f: I \longrightarrow \mathbb{R}$, at a pt $c \in I$, where I is an interval in $\mathbb{R}$, $I \neq pt$.
Then $c \in I \subseteq I'$. In the rest, we will assume $I \neq pt$. interval

6.1.6 **Thm** Let $f: I \longrightarrow \mathbb{R}$, $c \in I$, f be diff'ble at c.
Then f is continuous at c.

Proof: Since $I \neq pt \implies I \subseteq I'$, $c \in I'$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \cdot (x - c) + f(c) \right)$$

By Thm 5.1.13

$$= \underbrace{\lim_{x \to c} \frac{f(x) - f(c)}{x - c}}_{\text{exists}} \cdot \lim_{x \to c} (x - c) + \lim_{x \to c} f(c)$$

$$= f'(c) \cdot 0 + f(c) = f(c)$$

Remark:
- f diff'ble at $c \implies f$ cont at c.
- f cont at $c \not\Longrightarrow f$ diff'ble at c

  Ex $f(x) = |x|$, at 0.

- f diff'ble $\not\Longrightarrow f'$ is continuous

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- f' continuous $\implies f$ diff'ble

6.1.7 **Thm:** Let $f, g : I \to \mathbb{R}$, $I \neq pt$ (interval), $c \in I$.

Let f & g be diff'ble at c.

Then (i) $\forall k \in \mathbb{R}$, kf is diff'ble at c, $(kf)'(c) = kf'(c)$

(ii) $f+g$ is diff'ble at c, $(f+g)'(c) = f'(c) + g'(c)$

(iii) $f \cdot g$ is diff'ble at c, $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(iv) $\frac{f}{g}$ is diff'ble at c,

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c) \cdot g(c) - f(c) \cdot g'(c)}{(g(c))^2}$$

provided that $g(c) \neq 0$.

Remark: (Recall $g(x)$ diff'ble at c $\Rightarrow$ g cont at c

proof of IVT

$g(c) \neq 0 \Rightarrow \exists\, \delta > 0$ s.t.
$\forall x \in N(c, \delta) \Rightarrow g(x) \neq 0$)

(i), (ii) HW

(iv) after chain rule

Proof of (iii)

$$\lim_{x \to c} \frac{(f \cdot g)(x) - (f \cdot g)(c)}{x - c} = \lim_{x \to c} \frac{f(x) \cdot g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c}$$

Used Thm. 5.1.3, Thm 6.1.6, Def'n of diff'ble

$$= \lim_{x \to c} f(x) \cdot \frac{g(x) - g(c)}{x - c} + \lim_{x \to c} g(c) \cdot \frac{f(x) - f(c)}{x - c}$$

$$= f(c) \cdot g'(c) + g(c) \cdot f'(c)$$

#

CHAIN RULE Let $I \xrightarrow{f} J \xrightarrow{g} \mathbb{R}$

where I, J are intervals, $\neq$ pt. Let $c \in I$.

Let f be diff'ble at c,

g be diff'ble at $f(c)$

Then

(i) $g \circ f$ is diff'ble at c, and

(ii) $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

CAUTION The following approach/argument is **FAULTY**

If $x \neq c$
$$\frac{g(f(x)) - g(f(c))}{x - c} \overset{?}{=} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}$$

is 0 only at $x = c$

if $y = f(x)$ NO

$$\frac{g(y) - g(f(c))}{y - f(c)}$$

as $y \to f(c)$, $y \neq f(c)$ ok $\to g'(f(c))$

$\downarrow$ as $x \to c$: $f'(c)$

If f is not 1-1 about c then $f(x) = f(c)$ is possible for many values of $\underline{\underline{x \neq c}}$.

Ex $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

f is diff'ble at 0. (on $\mathbb{R}$ actually)

$x_k = \frac{1}{k\pi} \longrightarrow 0$, $\frac{1}{k\pi} \neq 0$

and $f\left(\frac{1}{k\pi}\right) = 0 = f(0)$

In other words:

$x \neq c \not\Rightarrow f(x) \neq f(c)$,

and one may be dividing by $0 = f(x) - f(c)$ even when $x \neq c$.

*** Proof of Chain Rule

$\odot$ Let $h(y) = \begin{cases} \dfrac{g(y) - g(f(c))}{y - f(c)} & \text{if } y \neq f(c) \\ g'(f(c)) & \text{if } y = f(c) \end{cases}$

on J

$\odot$ $h(y)\cdot(y - f(c)) = g(y) - g(f(c))$

($y = f(c) \Rightarrow$ both sides are 0)

$y \neq f(c)$ comes from above

Let $y = f(x)$

$\odot$ $h(f(x))[f(x) - f(c)] = g(f(x)) - g(f(c))$

$\odot$ $\lim_{x\to c} \dfrac{g(f(x)) - g(f(c))}{x - c} = \lim_{x\to c} \dfrac{h(f(x))\cdot[f(x) - f(c)]}{x - c}$

(**) $= \underbrace{\lim_{x\to c} h(f(x))}_{} \cdot \underbrace{\lim_{x\to c} \dfrac{f(x) - f(c)}{x - c}}_{f'(c)} = g'(f(c))\cdot f'(c).$

Want To show: $g'(f(c))$

Need Claim: $\lim_{x\to c} h(f(x)) = g'(f(c))$

(*) $\odot$ $\lim_{y\to f(c)} h(y) = \lim_{y\to f(c)} \dfrac{g(y) - g(f(c))}{y - f(c)} = g'(f(c)) = h(f(c))$

since g diffble at f(c)

h is continuous at f(c).

$\odot$ $g'(f(c)) \overset{\circledast}{=} h(f(c)) = h(\lim_{x\to c} f(x)) = \lim_{x\to c} h(f(x))$

f cont at c, $\lim_{x\to c} f(x) = f(c)$

use Thm 5.2.12

h is continuous at $f(c)$.

This proves claim, hence the Chain Rule by $\circledast\circledast$.

6.1 Exc #15 p.247

Prove quotient rule from Chain Rule + Product Rule.

$\odot$ $\frac{1}{g(x)} = r(g(x))$

Let $r(x) = \frac{1}{x}$. $\quad r'(x) = -\frac{1}{x^2}$

Chain Rule $\left(\frac{1}{g(x)}\right)' = r'(g(x))\cdot g'(x) = -\frac{1}{g(x)^2}\cdot g'(x)$

$\odot$ $\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' = f'\cdot\frac{1}{g} + f\cdot\left(\frac{1}{g}\right)'$

$= f'\cdot\frac{1}{g} + f\cdot\frac{-g'}{g^2}$

$= \frac{f'\cdot g - f\cdot g'}{g^2}$