Nov 26, 2018

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**Corollary:** (of Ext. V. Th & Int V. Th)

If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then

$$f([a,b]) = [m, M].$$

Emphasize ( $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ )

**Proof:** Ext V. Thm $\Rightarrow f$ must attain its max/min values
i.e. $\exists x_1, x_2 \in [a,b]$ s.t.
$\forall x \in [a,b]$,

$$\min \text{ of } f = m = f(x_1) \leq f(x) \leq f(x_2) = M = \max \text{ of } f$$

$$f([a,b]) \subseteq [m, M].$$

IntV Thm $\Rightarrow$ $\forall k$ between $m$ & $M$ $\exists c \in [a,b]$ $f(c) = k$
$\forall k$ " $m$ & $M$, $k \in f([a,b])$
$$f([a,b]) = [m, M].$$

**Remark:** If $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is continuous
and $I$ is an interval of $\mathbb{R}$, $I \subseteq D$,
then $f(I)$ is (another) interval.
Actually it is equivalent to Intermediate V. Thm.

Caution: Other than the type $[a,b]$,
$f(I)$ may not be of the type as $I$.
Ex $f(x) \equiv c$ $f(\text{any interval}) = [c, c]$.

5.3 Exc. 3 p 220

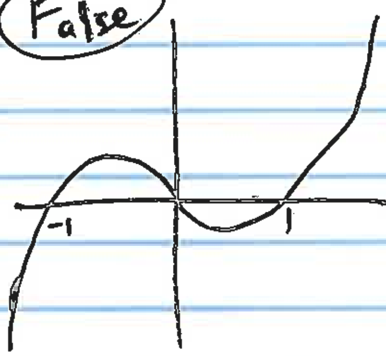
a) False

$f(x) \equiv 1 \qquad \underset{\text{open}}{f(\mathbb{R})} = \underset{\text{not open}}{\{1\}}$

b) False

$f(x) = e^x \qquad \underset{\text{closed}}{f(\mathbb{R})} = \underset{\text{not closed}}{(0, \infty)}$

c) False




$f(x) = x(x^2 - 1)$

$f(\underbrace{(-\infty, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, \infty)}_{\text{not open}}) = \underset{\text{open}}{\mathbb{R}}$

(F) d) $f(x) \equiv 1$

$f(\text{any set}) = \{1\}$ ← closed, compact, bounded, finite

choose any not closed set

(h) False choose any infinite set

(F) e) choose any non-compact set, $\{1\}$ is compact

(F) f) $f(x) = \tan^{-1} x \, : \, \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$

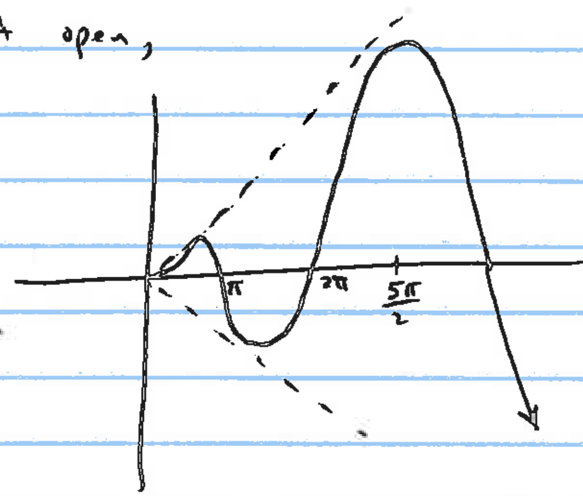
(g) & (i) True

Exc 5.3.3 (j) D is interval not open
$\overset{?}{\Rightarrow}$ f(D) " " not open } False

Ans NO

We Want D interval not open, but f(D) open.

$f(x) = x \sin x$




not open. open.

$f([0,\infty)) = \mathbb{R}$

Actually

$f([a,\infty)) = \mathbb{R}$ for any $a \in \mathbb{R}$.

End of Chap 5 (5.3).

## Chap VI Differentiation

**Defn.** Let $f : I$ (an interval in $\mathbb{R}$) $\longrightarrow \mathbb{R}$, $c \in I$
$f$ is called differentiable at $c$ if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{exists \& is finite.}$$

We denote this limit by $f'(c)$, when it exists.

- If $S$ is the set $\{c \in I \mid f'(c)$ is defined & finite$\}$

then $f' : S \to \mathbb{R}$ is called the derivative of $f$.

**Ex** $f(x) = 6x^2 + 7x - 1$

$$\lim_{x \to c} \frac{f(c) - f(x)}{c - x} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c} \frac{(6x^2 + 7x - 1) - (6c^2 + 7c - 1)}{x - c}$$

$$= \lim_{x \to c} \frac{6(x^2 - c^2) + 7(x - c)}{x - c}$$

$$= \lim_{x \to c} 6(x + c) + 7 = 12c + 7 = f'(c)$$

Ex: $g(x) = \frac{1}{\sqrt{x}} : (0,\infty) \to \mathbb{R}$.

$x, c > 0$

$$\lim_{x\to c} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{c}}}{x-c} = \lim_{x\to c} \frac{\frac{\sqrt{c}-\sqrt{x}}{\sqrt{x}\sqrt{c}}}{x-c} = \lim_{x\to c} \frac{\sqrt{c}-\sqrt{x}}{\sqrt{x}\sqrt{c}(x-c)}$$

$x \neq c$

$(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})$

$$= \lim_{x\to c} \frac{-1}{\sqrt{x}\sqrt{c}(\sqrt{x}+\sqrt{c})} = \frac{-1}{2c\sqrt{c}}.$$

$g' : (0,\infty) \to (-\infty, 0)$

6.1.3: <u>Thm:</u> Let $f : I \to \mathbb{R}$, $c \in I$.

$f$ is diff'ble at $c \iff \forall$ sequence $(x_n)$ in $I$,

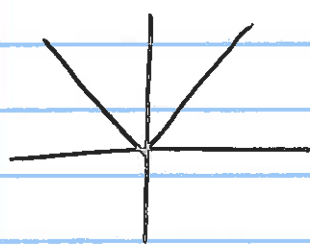
$x_n \neq c$, $x_n \to c$,

$\frac{f(x_n)-f(c)}{x_n - c}$ is convergent

Thm 5.1.8

<u>Remark:</u> Useful to show $f'(c)$ DNE.

Ex #1 $f = |x|$ is not diff'ble at 0, by using Thm 6.1.3



$x_n = \left(\frac{1}{n}\right)(-1)^n \to 0$

$$\frac{f(x_n)-f(0)}{x_n - 0} = \frac{\left|\frac{1}{n}(-1)^n\right| - 0}{\frac{1}{n}(-1)^n - 0} = (-1)^n$$

$(-1)^n$ is a divergent sequence.

Thm 6.1.3 $\Rightarrow$ $f'(0)$ DNE.

(Ex)

(ii) $f(x) = \begin{cases} x \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(i) $g(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Assume basic diff'n rules (to be done in Wed 11/28) from Calc I

$g'(x) = ?$

Case 1 $x \neq 0$

$$g'(x) = 2x \sin\frac{1}{x} + x^2 \cdot \left(\cos\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$= 2x \sin\frac{1}{x} - \cos\frac{1}{x}.$$

Case 2 $g'(0) = ?$

$$\lim_{x \to 0} g'(x) = \lim_{x \to 0} \underbrace{2x \sin\frac{1}{x}}_{0} - \underbrace{\cos\frac{1}{x}}_{\text{DNE}} \quad \text{DNE.}$$

0 since $2x \to 0$, $|\sin\frac{1}{x}| \le 1$

$$\lim_{\substack{x \to 0 \\ x \neq 0}} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x - 0} = \lim_{x \to 0} x \sin\frac{1}{x} = 0$$

$\left(\forall \varepsilon > 0\ \exists \delta = \varepsilon > 0\ \forall x \in \mathbb{R}\quad \left|x \sin\frac{1}{x} - 0\right| \le |x| < \delta = \varepsilon.\right)$

$$g'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$g'(x) : \mathbb{R} \longrightarrow \mathbb{R}$, defined everywhere on $\mathbb{R}$.

but $\lim_{x \to 0} g'(x) \neq g'(0) = 0$

$g'(x)$ is not continuous at 0.

Compare $h(x) = |x| : \mathbb{R} \to \mathbb{R}$

$$h'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{undef} & \text{if } x = 0 \end{cases}$$

$h' : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ (Actually continuous since $0 \notin$ domain of $h'$.)

$\lim_{x \to 0} h'(x)$ DNE $\longleftarrow$ can't compare to $h'(0)$ which DNE

(ii)

$$f(x) = \begin{cases} x \cdot \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$x \neq 0$ $\quad f'(x) = 1 \sin\frac{1}{x} - x\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

$$= \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$$

$f'(0)$ DNE ??

PTO

Let $x_n = \dfrac{1}{\left(\frac{\pi}{2} + n\pi\right)} \xrightarrow[n\to\infty]{x_n \neq 0} 0$

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{x_n \sin\frac{1}{x_n} - 0}{x_n - 0}, \quad (x_n \neq 0)$$

$$= \sin\frac{1}{x_n} = \sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^n$$

$(-1)^n$ diverges; Thm 6.1.3 $\Rightarrow$ $f'(0)$ DNE.