***** Thm: $f: D \to \mathbb{R}$ continuous, D compact $\implies f(D)$ compact

has a very important consequence:

*** Thm (Extreme Value Thm) (Max-Min Thm)

Let $f: D \to \mathbb{R}$ be continuous,

& Let D be closed & bounded $\subseteq \mathbb{R}$

Then $\exists\, x_1, x_2 \in D$ s.t. $\forall x \in D$,

$$f(x_1) \le f(x) \le f(x_2).$$

In other words, f must attain its max/min on D, when D is compact & f is continuous.

$\subseteq \mathbb{R}$

* Lemma 2: Let A be closed & bounded subset of $\mathbb{R}$.

Then $\max(A)$ & $\min(A)$ exist.

Proof: A is bounded $\subseteq \mathbb{R} \implies \sup A$ exists & $\in \mathbb{R}$

& $\inf A$ exists & $\in \mathbb{R}$

Let $L = \sup A$. $\forall n \in \mathbb{N}$ $L - \frac{1}{n}$ is not an upper bound for A.

$$\exists\, x_n \in A \qquad L - \tfrac{1}{n} < x_n \le L$$

as $n \to \infty$: $L - \frac{1}{n} \to L$, $L \to L$

By squeeze Thm $x_n \to L$.

$x_n \in A$, $x_n \to L$, A is closed

(Nov 14: Lemma I) $\implies$ $L \in A$

$L = \sup A = \max A$. #.

Similar proof for $\min = \inf$ for compact subsets of $\mathbb{R}$.

<u>Proof of</u> Extreme Value Thm.

$f: D \to \mathbb{R}$ continuous

D is closed & bounded, hence compact (in $\mathbb{R}$)

***** Thm $\Rightarrow$ $f(D)$ is compact

Lemma 2 $\quad \max f(D) = L$ exists

$\min f(D) = K$ exists.

$L \in f(D)$, $\exists\, x_2 \in D$ s.t. $f(x_2) = L$

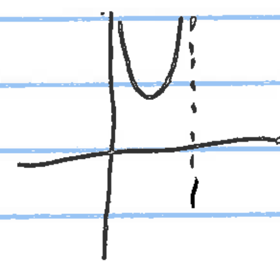
$K \in f(D)$, $\exists\, x_1 \in D$ s.t. $f(x_1) = K$.

$\forall x \in D$, $f(x) \in f(D)$

$$f(x_1) = K = \min f(D) = \inf f(D) \le f(x) \le \sup f(D) = \max f(D) = L = f(x_2)$$

$\forall x \in D \quad f(x_1) \le f(x) \le f(x_2)$.

Ex (i) $f(x) = \frac{1}{x} + \frac{1}{1-x} : (0,1) \to \mathbb{R}$



f cont, domain is bdd

domain is not closed

f unbounded.

(ii) $g(x) = x^2 : [0,4)$ domain bdd, not closed

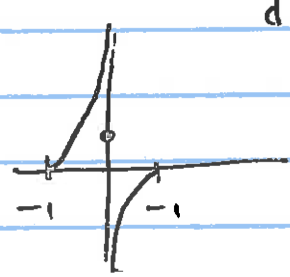
$g(x)$ is bounded $\quad 0 \le g(x) < 16$

$16 \ne f(x_2)$ for any $x_2 \in [0,4)$

(iii) $f(x) = x : \mathbb{R} \to \mathbb{R}$

domain closed, domain unbounded, f is unbounded

(iv)



Domain compact f not continuous

f not bounded.

******* INTERMEDIATE VALUE THM**

Recall $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$.

Let $f : [a,b] \to \mathbb{R}$ be continuous.

$\forall k$ between $f(a)$ and $f(b)$ $\exists c \in [a,b]$ s.t. $f(c) = k$.

Remark: False in $\mathbb{Q}$ $\quad f(x) = x^2 - 2 : \mathbb{Q} \to \mathbb{Q}$.

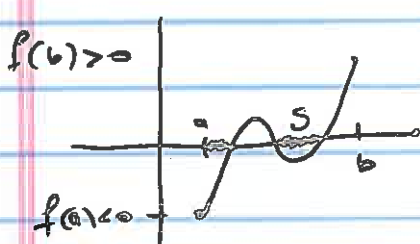
$$f(0) = -2$$
$$f(2) = 2$$

$\exists$ no $c \in \mathbb{Q}$ s.t. $f(c) = 0$.

Proof **Crucial Case** $\quad f(a) < 0 < f(b)$

$\Rightarrow \exists c \in (a,b)$ s.t. $f(c) = 0$.

Let $S = \{x \in [a,b] \mid f(x) \le 0\} \subseteq \mathbb{R}$



$S \ne \emptyset$, since $a \in S$.

S is bounded since $S \subseteq [a,b]$.

Completeness Axiom

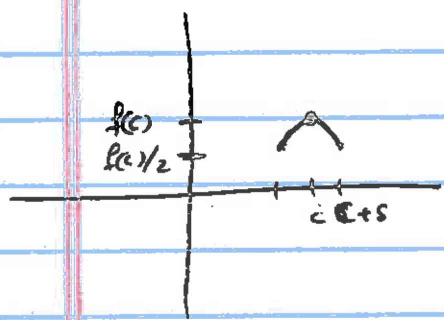
$\Rightarrow \exists c = \sup S \in \mathbb{R}$.

$a \le c \le b$

$a \in S$ — least upper bd for S. — an upper bd for S.

Want to show $f(c) = 0$.

Suppose $f(c) > 0$. $\quad c > a$ since $f(a) < 0$.



For $\varepsilon = \dfrac{f(c)}{2} > 0$

$\exists\, \delta > 0$ s.t.

$\forall x \in D$, $\quad |x - c| < \delta \implies$

$D = [a,b]$. $\qquad |f(x) - f(c)| < \varepsilon = \dfrac{f(c)}{2}$

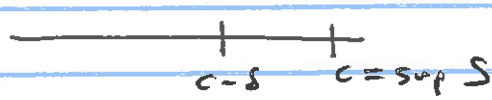
$$-\frac{f(c)}{2} = -\varepsilon < f(x) - f(c) < \varepsilon = \frac{f(c)}{2}$$

$$0 < \frac{1}{2} f(c) = f(c) - \frac{f(c)}{2} < f(x) < f(c) + \frac{f(c)}{2} = \frac{3}{2} f(c)$$

$\forall x \qquad c - \delta < x \le c \qquad f(x) > \dfrac{f(c)}{2} > 0$

recall $S = \{x \in [a,b] \mid f(x) \le 0\}$

$c = \sup S$

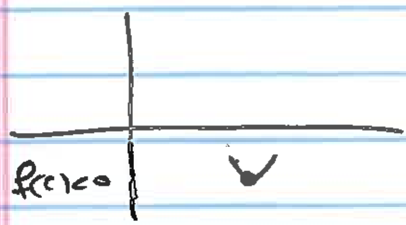


$\underline{(c-\delta, c]}$ can't belong to $S \implies c \ne \sup S$.

contradiction

<u>Conclusion</u> $f(c) \le 0$

Suppose $f(c) < 0$. $c < b$ since $f(b) > 0$



For $\varepsilon = \dfrac{|f(c)|}{2} > 0$

$\exists\, \delta' > 0$ s.t.

$\forall x \in D$, $|x-c| < \delta' \Longrightarrow$

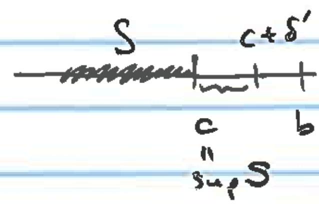
$|f(x) - f(c)| < \varepsilon = \dfrac{|f(c)|}{2}$.

$$\frac{f(c)}{2} = -\varepsilon < f(x) - f(c) < \varepsilon = -\frac{f(c)}{2}$$

$$\frac{3f(c)}{2} < f(x) < \frac{f(c)}{2} < 0$$

$$\forall x \quad \left.\begin{array}{c} c \le x < c+\delta' \\ x \in D \end{array}\right\} \quad f(x) < \frac{f(c)}{2} < 0.$$

Recall $S = \{\, x \in [a,b] \mid f(x) \le 0 \,\}$



$[c, c+\delta'') \subseteq S$

$\delta'' = \min(\delta', b-c)$

$c \neq \sup S$.

Contradiction

$f(c) > 0$ and $f(c) < 0$ are not possible, $c \in [a,b]$,

$f(c)$ defined, $\in \mathbb{R}$.

$\Longrightarrow f(c) = 0$

PTS General Case

Let $k \in \mathbb{R}$ be given between $f(a)$ & $f(b)$.

General Case 1 $\quad f(a) < k < f(b)$

Let $g(x) = f(x) - k$. g is continuous.

$g(a) < 0 < g(b)$

$\exists c \in (a,b)$ s.t. $g(c) = 0 = f(c) - k$

$f(c) = k$.

Case 1' $\quad$ if $f(a) = k$ take $c = a$

Case 1'' $\quad$ if $f(b) = k$ take $c = b$.

Case 2 $\quad f(b) < k < f(a)$, $a < b$.

Let $h(x) = k - f(x)$. h is continuous

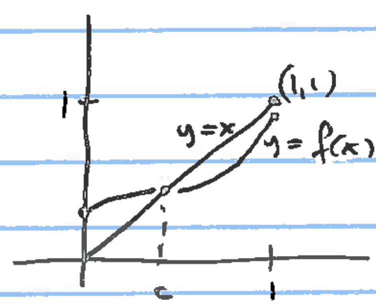
$h(a) < 0 < h(b)$

$\exists c \in (a,b)$ s.t. $h(c) = 0 = k - f(c)$ #

5.3 ~~Ex #7~~ P

Let $f(x): [0,1] \to [0,1]$ be continuous.

then $\exists c \in [0,1]$ s.t. $f(c) = c$



$0 \le f(x) \le 1$

Let $g(x) = f(x) - x$

$g(0) = f(0) - 0 = f(0) \ge 0$

$g(1) = f(1) - 1 \le 0$

$g(1) \le 0 \le g(0)$

By Int V. Thm $\exists c$ s.t. $g(c) = 0$

$g(c) = f(c) - c = 0$

$f(c) = c$

Sol^n of the General Case is almost the same:

$f(x): [a,b] \to [a,b]$. $\quad g(x) = f(x) - x$

$a \le f(x) \le b \quad g(a) = f(a) - a \ge 0$

$g(b) = f(b) - b \le 0 \quad \exists c \in [a,b]$

$g(c) = 0$

$f(c) - c = 0$.