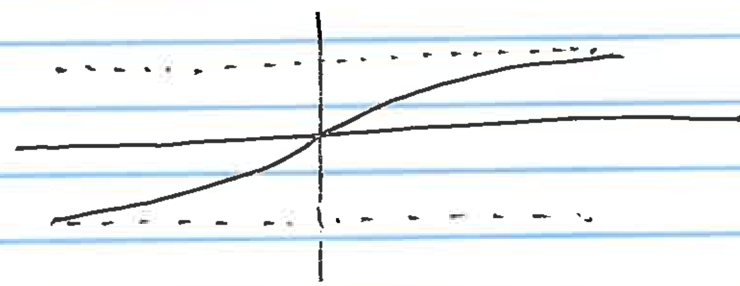
5.3

Defn A function $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, is called bounded if $\exists M \in \mathbb{R}$ s.t. $\forall x \in D$ $|f(x)| \le M$.
(or equivalently $f(D) \subseteq [-M, M]$ for some $M \in \mathbb{R}$)
(or equivalently $f(D)$ is a bounded set.)

Ex ① $f(x) = \sin x : \mathbb{R} \to \mathbb{R}$. is a bounded function.
$\forall x \in \mathbb{R}$ $|\sin x| \le 1$, $f(\mathbb{R}) = [-1, 1]$.
(domain $\mathbb{R}$ is not bounded)

② $g(x) = \tan^{-1} x : (-\infty, \infty) \to (-\infty, \infty)$



$\tan^{-1}(\mathbb{R}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Overview

Let $f: D \to E$ be continuous in all below

$D \subseteq \mathbb{R}$, $E \subseteq \mathbb{R}$

$A \subseteq D$, A bounded $\not\Rightarrow$ $f(A)$ bounded

Ex. $f(x) = \frac{1}{x} : (0,1) \to \mathbb{R}$ $\quad f((0,1)) = (1, \infty)$

$A \subseteq D$, A is closed in $\mathbb{R}$ $\not\Rightarrow$ $f(A)$ closed in $\mathbb{R}$

Ex. $f(x) = \frac{1}{x} : [1, \infty) \to \mathbb{R}$ $\quad f([1,\infty)) = (0,1]$

$A \subseteq D$, A is open in $\mathbb{R}$ $\not\Rightarrow$ $f(A)$ open in $\mathbb{R}$.

Ex. $g(x) = \sin x : \mathbb{R} \to \mathbb{R}$ $\quad \sin(\mathbb{R}) = [-1,1]$.

Theorem: $A \subseteq D$, A is $\underbrace{\text{compact}}_{\text{closed \& bounded in } \mathbb{R}}$ in $\mathbb{R}$ $\Longrightarrow$ $f(A)$ $\underbrace{\text{compact}}_{\text{closed, bounded in } \mathbb{R}}$ in $\mathbb{R}$.

Recall: Heine-Borel: $\forall A \subseteq \mathbb{R}$ (A is compact $\Longleftrightarrow$ A is closed & bounded)

$E = D = \mathbb{R}$, $B \subseteq E$, B closed $\Longrightarrow$ $f^{-1}(B)$ closed

$E = D = \mathbb{R}$, $B \subseteq E$, B is open $\Longrightarrow$ $f^{-1}(B)$ open

Thm. see Nov 12 p 4

$E = D = \mathbb{R}$, $B \subseteq E$ B is bounded $\not\Rightarrow$ $f^{-1}(B)$ bounded

Ex $f(x) \equiv 1 : \mathbb{R} \to \mathbb{R}$. $\quad f^{-1}(\{1\}) = \mathbb{R}$.

$E = D = \mathbb{R}$, $B \subseteq E$ B is compact $\not\Rightarrow$ $f^{-1}(B)$ compact

(closed & Bounded) $\qquad$ (closed & bounded)

THM: Let $f : D \to \mathbb{R}$ be s.t

(1) f is continuous on D, and

(2) D is closed & bounded $\subseteq \mathbb{R}$ (D is compact),

then

$f(D)$ is closed and bounded $\subseteq \mathbb{R}$ ($f(D)$ is compact)

Lemma I Let $A \subseteq \mathbb{R}$.

A is closed in $\mathbb{R}$

$\iff$ Every convergent sequence in A has a limit in A

i.e. $\left(\forall (s_n) \subseteq A,\ s_n \to s_0 \implies s_0 \in A \right)$

Proof: Recall $\left[\begin{array}{l} A \text{ is closed} \iff Bd\,A \subseteq A \\ \qquad\qquad\quad \iff A' \subseteq A \\ \qquad\qquad\quad \iff cl\,A = A \cup A' = A \end{array} \right]$ Oct 26 p 2

$\implies$: Assume A is closed

Let a sequence $(s_n) \subseteq A$ be given s.t $s_n \to s_0$.

Case 1 $s_n = s_0$ for some n, $s_0 = s_n \in A$. ✓

Case 2 $s_n \neq s_0$ for all n.

Let $\varepsilon > 0$ be arbitrary $\exists N\ \forall n \geq N\ \ 0 < |s_n - s_0| < \varepsilon$ ($s_n \neq s_0$)

$s_n \in N^*(s_0, \varepsilon) \cap A \neq \emptyset$.

$s_0 \in A'$

$A' \subseteq A$ since A is closed

$s_0 \in A$.

$\impliedby$: we will prove contrapositive.

<u>Want</u>: $\left[A \text{ is not closed} \Rightarrow \exists \text{ sequence } (x_n) \subseteq A,\ x_n \to s_0 \text{ but } s_0 \notin A. \right]$

Start with A is not closed.

$\text{bd}\, A \not\subseteq A.$ (defn)

$\exists\, s_0 \in \text{bd}\, A,\ s_0 \notin A$

def of bd. $\forall\, \underset{n \in \mathbb{N}}{\varepsilon = \tfrac{1}{n}},\quad N(s_0, \tfrac{1}{n}) \cap A \neq \emptyset$

$\exists\, x_n \in N(s_0, \tfrac{1}{n}) \cap A$

$-\tfrac{1}{n} < x_n - s_0 < \tfrac{1}{n}, \quad x_n \in A.$

as $n \to \infty$: $-\tfrac{1}{n} \to 0$, $\tfrac{1}{n} \to 0$

$x_n - s_0 \to 0$ by squeeze Thm.

$x_n \to s_0 \notin A$

but (x_n) is a sequence in A. #

✻✻✻✻✻✻ **Proof of Thm**
$$\left.\begin{cases} f: D \to \mathbb{R},\ D \subseteq \mathbb{R}. \\ D \text{ compact and} \\ f \text{ continuous} \end{cases}\right\} \Longrightarrow f(D) \text{ is compact}$$

Recall Heine Borel Thm

$$\forall D \subseteq \mathbb{R} \ (\ D \text{ compact} \Longleftrightarrow D \text{ closed \& bounded}\ )$$

Given $f: D \to \mathbb{R}$ continuous
D closed & bounded in $\mathbb{R}$

Want $f(D)$ closed & bounded in $\mathbb{R}$

① To show $f(D)$ is bounded

To prove by contradiction. Suppose not; suppose $f(D)$ is unbounded.

$$\text{not } (\underbrace{\exists M \in \mathbb{R}\ \forall x \in D,\ |f(x)|}_{\text{bounded}} \le M)$$

$$\Longrightarrow \forall M \in \mathbb{R}\ \exists x_M \in D \quad (|f(x_M)| > M)$$

$$\Longrightarrow \forall n \in \mathbb{N}\ \exists x_n \in D \quad |f(x_n)| > n.$$

(x_n) is a sequence in D.

D is closed & bounded

Recall Bolzano-Weierstrass p⑤ Oct 31, 2018 & p⑤ Oct 29 Thm 2

Hence (x_n) has a convergent subsequence (x_{n_k})

$$x_{n_k} \longrightarrow x_0$$

D is closed, $\underset{\text{Today}}{\text{Lemma I}} \Rightarrow x_0 \in D$.

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$$

For $\varepsilon = 1$ $\exists K$ $\forall k \geq K$ $\quad |f(x_{n_k}) - f(x_0)| < 1.$

$$|f(x_{n_k})| - |f(x_0)| < |f(x_{n_k}) - f(x_0)| < 1$$

$$k \leq n_k < |f(x_{n_k})| < 1 + |f(x_0)|$$

section 4.4

as $k \to \infty$ it is not possible to have all

$$k < \underbrace{1 + |f(x_0)|}_{\text{a fixed value}} \quad \text{(by Archimedean Princ.)}$$

Contradiction.

Hence $f(D)$ is bounded.

(2) To show $f(D)$ is closed.

We will use Lemma I (today).

Let (y_n) be any sequence in $f(D)$

s.t. $y_n \to y_0$. Want to show $y_0 \in f(D)$.

$y_n \in f(D)$, $y_n = f(x_n)$ for some $x_n \in D$.

D is closed & bounded

p5 Oct 29
p5 Oct 31, Thm 2 $\quad \exists$ convergent subsequence (x_{n_k}) of (x_n)

s.t. $x_{n_k} \to x_0$.

D closed: $x_0 \in D$ by Lemma I.

$$y_0 = \lim y_n = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k}) = f(x_0) \in f(D) \quad \#$$

Hence $f(D)$ is closed & bounded, $\subseteq \mathbb{R}$

$f(D)$ is compact by Heine Borel Thm.

* * * * * Thm: $f : D \to \mathbb{R}$ continuous, D compact $\Longrightarrow$ $f(D)$ compact

has a very important consequence:

* * * Thm (Extreme Value Thm)

Let $f : D \to \mathbb{R}$ be continuous,

& Let D be closed & bounded.

Then $\exists\, x_1, x_2 \in D$ s.t. $\forall x \in D$,

$$f(x_1) \leq f(x) \leq f(x_2).$$

In other words f must attain its max/min