Nov 12, 2018

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5.2 Continue

Ex Polynomial functions

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad n \in \mathbb{N},\ a_i \in \mathbb{R}$$

Rational functions $R(x) = \dfrac{P(x)}{Q(x)}$ where

$P(x), Q(x)$ are polynomials.

Every polynomial function $P(x): \mathbb{R} \longrightarrow \mathbb{R}$ is continuous:

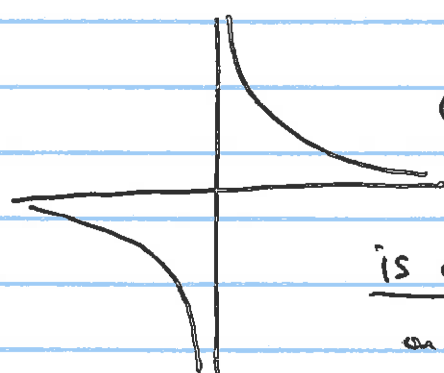
$$\lim_{x \to c} P(x) = P(c) \quad (\text{proved it})$$

$$R(x) = \frac{P(x)}{Q(x)} : D = \{x \in \mathbb{R} \mid Q(x) \neq 0\} \longrightarrow \mathbb{R}.$$

$$\lim_{x \to c} R(x) = \frac{P(c)}{Q(c)} \quad \text{if } Q(c) \neq 0.$$

$R(x)$ is continuous on $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$.

Ex $R(x) = \dfrac{1}{x} : \mathbb{R} - \{0\} \to \mathbb{R}$

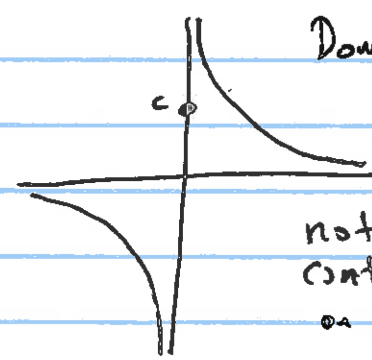


$0 \notin$ Domain

is continuous on $\underbrace{\mathbb{R} - \{0\}}$

$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ c & \text{if } x = 0 \end{cases}$

Domain $= \mathbb{R}$



not continuous on $\mathbb{R}$.

Thm: Let $f, g : D \to \mathbb{R}$ be both continuous at $c \in D$
Then
1. $f + g$ is continuous at c
2. $f \cdot g$ " " " "
3. $k \cdot f$ " " " " for $k \in \mathbb{R}$
4. f/g " " " " if $g(c) \neq 0$.

Proof: if $c \notin D'$, every function is continuous at isolated pts.

If $c \in D'$, $x_n \to c$

$$\left.\begin{array}{l} f(x_n) \to f(c) \\ g(x_n) \to g(c) \end{array}\right\} \quad (f+g)(x_n) = f(x_n) + g(x_n) \to f(c) + g(c)$$

by Thm 5.1.8

$$\lim_{x \to c} (f+g)(x) = f(c) + g(c)$$

Thm:

Let $D \xrightarrow{f} E \xrightarrow{g} \mathbb{R}$, $D \subseteq \mathbb{R}$, $E \subseteq \mathbb{R}$, f be continuous at $c \in D$, g be " at $f(c) \in E$.

Then: $g \circ f$ is continuous at c.

Proofs: (I) if $c \notin D'$, there is nothing to prove. If $c \in D'$: Let $x_n \to c$ be any sequence in D.

Thm 5.2.2 $\Rightarrow$ $y_n = f(x_n) \to f(c)$ since f continuous at c.

" " $\Rightarrow$ $g(y_n) \to g(f(c))$ " g continuous at $f(c)$

$(g \circ f)(x_n) = g(f(x_n)) \to g(f(c)) = g \circ f(c)$.

$g \circ f$ continuous at c. by Thm 5.2.2

Another Proof:

I Continuity of g at $d = f(c)$. $d = f(c)$

$\forall \varepsilon > 0 \; \exists \eta > 0 \; \forall y \in E \; (\; |y - d| < \eta$
$\Longrightarrow |g(y) - g(d)| < \varepsilon \;)$

Given $\eta > 0 \; \exists \delta > 0 \; \forall x \in D$
(continuity of f at c) $(\; |x - c| < \delta$
$\Longrightarrow |f(x) - f(c)| < \eta. \;)$

Combining these two sentences

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D. \; |x - c| < \delta$
$\Longrightarrow |\underbrace{f(x)}_{= y} - \underbrace{f(c)}_{= d}| < \eta$

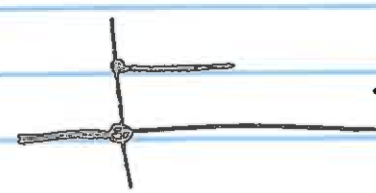
$\Longrightarrow |g(f(x)) - g(f(c))| < \varepsilon. \;)$

By Def of continuity, $g \circ f$ is continuous at c.

Exc. 5.2. #1 p212-213

1 a) T (Def.)

b) False



$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

$f(\mathbb{R}) = \{0, 1\}$ bdd.

c) True

d) False (You need $x_n \to c \Longrightarrow f(x_n) \to f(c)$)
has $f(x_n) \to f(c) \Longrightarrow x_n \to c$

e) $f(x) \equiv 1$ $f(U \cap D) = \{1\} \neq V$ open
$f: \mathbb{R} \to \mathbb{R}$ not open

This is a combination of <u>Exc</u> 5.2.16 + Corollary 5.2.15.
(No need to know Thm 5.2.14)

(*)(*)(*) <u>Thm:</u> Let $f: \underset{\underset{\text{domain}}{D}}{\mathbb{R}} \longrightarrow \underset{\underset{\text{codomain}}{C}}{\mathbb{R}}$

Then the following 3 statements are equivalent

(a) f is continuous on $\mathbb{R}$,

(b) $\forall$ open set $U \subseteq \mathbb{R} = C$, $f^{-1}(U)$ is open in $D = \mathbb{R}$.

$\boxed{\text{In topological spaces, this is the defn of continuity}}$

(c) $\forall$ closed set $H \subseteq \mathbb{R} = C$, $f^{-1}(H)$ is closed in $D = \mathbb{R}$

<u>Proof:</u>

$$\begin{tikzcd} & a \arrow[dr, Rightarrow] & \\ b \arrow[ur, Rightarrow] & & c \arrow[ll, Rightarrow] \end{tikzcd}$$

(c) $\Longrightarrow$ (b)

Assume (c), to prove (b)

Let $U \subseteq C$ be a given open set in $\mathbb{R}$

Let $H = \mathbb{R} - U$, it is closed (complement of open set)

$f^{-1}(H)$ is closed. by hypothesis (c)

$$f^{-1}(H) = f^{-1}(\mathbb{R} - U) = \underbrace{f^{-1}(\mathbb{R})}_{\mathbb{R}} - f^{-1}(U) = \mathbb{R} - f^{-1}(U)$$

$f^{-1}(U)$ is open by being the complement of a closed set, $f^{-1}(H)$.

$\forall U$ open $\subseteq \mathbb{R}$, $f^{-1}(U)$ is open.

(b) $\Longrightarrow$ (a)

(a): Want $\left[\begin{array}{l}\text{Let } c \in \mathbb{R}, \text{ Let } \varepsilon > 0 \text{ be given} \\ \exists \delta > 0 \ \forall x \in \mathbb{R} \left(|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \right)\end{array}\right.$

Let $c \in \mathbb{R}$, $\varepsilon > 0$ be given.

Let $U = N(f(c), \varepsilon)$, which is an open set.

$f^{-1}(U)$ is open in $\mathbb{R}$

$c \in \underbrace{f^{-1}(U)}_{\text{call it } V}$ since $f(c) \in U$

$c \in V$ which is open.

(Def^n of open) $\exists \delta > 0$ $N(c, \delta) \subseteq V$.

$f(N(c,\delta)) \subseteq f(V) = f(f^{-1}(U)) \subseteq U$

Combining all:

Given $c \in \mathbb{R}$, $\forall \varepsilon > 0$ $\exists \delta > 0$ $f(N(c,\delta)) \subseteq N(f(c), \varepsilon))$.

$x \in N(c,\delta) \Longleftrightarrow |x-c| < \delta$

Given $c \in \mathbb{R}$, $\forall \varepsilon > 0 \ \exists \delta > 0$, $\forall x \in \mathbb{R} \left(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \right)$

$f(x) \in N(f(c), \varepsilon)$

(a) $\Longrightarrow$ (c)

given f is continuous on $\mathbb{R}$:

$\forall c \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}\ (|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

Want $\forall$ closed set $H \subseteq \mathbb{R} =$ codomain,

$f^{-1}(H)$ is closed in $D = \mathbb{R}$.

Proof: Let $H \subseteq \mathbb{R}$, H be closed.

Let $E = f^{-1}(H)$.

If $E = \emptyset$ or $E' = \emptyset$, then $E = f^{-1}(H)$ is closed, which finishes the proof.

Assume $E' \neq \emptyset$ $(\Rightarrow E \neq \emptyset)$

Let $c \in E'$, be an arbitrary pt.

$\exists (s_n) \subseteq E$, $s_n \xrightarrow{\neq} c$ [Since $\forall \varepsilon > 0$ $N^*(c, \varepsilon) \cap E \neq \emptyset$ OR $\forall n \in \mathbb{N}$, $\underbrace{N^*(c, \tfrac{1}{n}) \cap E \neq \emptyset}_{\text{choose } s_n \text{ in it.}}$

$s_n \in E = f^{-1}(H)$

$f(s_n) \in H$.

$f(s_n) \to f(c)$ hypothesis (a), Thm 5.2.2

Claim $f(c) \in H \cup H'$.

If $f(s_n) = f(c)$ for some n, then $f(c) \in H$

If $f(s_n) \neq f(c)$ for all n, $f(s_n) \to f(c)$

def^n of H' $\Longrightarrow$ $f(c) \in H'$

$H = H \cup H'$ since H is closed.

$f(c) \in H$

$c \in f^{-1}(H) = E$.

We proved for any $c \in E'$, we have $c \in E$.

Hence $E' \subseteq E$.

$f^{-1}(H) = E$ is closed by def^n