Nov 9, 2018

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5.1 Exc 7c, 8b, 8a. page 204

7c prove that $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ if $c \geq 0$, $x \geq 0$.

Proof Case 1 $c = 0$.

We want $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in [0, \infty) \; \left(0 < |x - 0| < \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon \right)$

need $\delta = \varepsilon^2$ $\qquad$ $\sqrt{0}$

$\forall \varepsilon > 0 \; \exists \delta = \varepsilon^2 > 0 \; \forall x \in [0, \infty) \; \left(0 < |x - 0| < \delta \right.$

$\Rightarrow |x| < \delta = \varepsilon^2$

$\Rightarrow |\sqrt{x} - 0| < \sqrt{\varepsilon^2} = \varepsilon$.

Case 2 $c > 0$

$\forall \varepsilon > 0 \; \exists \delta = \varepsilon \sqrt{c} > 0, \; \forall x \in [0, \infty)$

$\left(0 < |x - c| < \delta \Rightarrow \right.$

$$|\sqrt{x} - \sqrt{c}| = \left| \frac{(\sqrt{x} - \sqrt{c}) \cdot (\sqrt{x} + \sqrt{c})}{\sqrt{x} + \sqrt{c}} \right| = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}}$$

$$\frac{|x - c|}{\sqrt{c}} < \frac{\delta}{\sqrt{c}} = \varepsilon.$$

7.c $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ done

Exc 8.b

Prove $\lim_{x \to c} f(x) = L \implies \lim_{x \to c} \sqrt{f(x)} = \sqrt{L}$.

$f(x) \geq 0$

$L \geq 0$

Let s_n be any sequence $s_n \xrightarrow{\neq} c$.

Thm 5.1.8. : $\lim_{n \to \infty} f(s_n) = L$

Let $d_n = f(s_n) \geq 0$. Then, $d_n \to L$.

Exc 7.c $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ $\lim_{n \to \infty} \sqrt{d_n} = \sqrt{L}$.

$$\sqrt{d_n} = \sqrt{f(s_n)} \longrightarrow \sqrt{L}$$

By using Thm 5.1.8. $\lim_{x \to c} \sqrt{f(x)} = \sqrt{L}$.

Exc 8a Prove $\underbrace{\lim_{x \to c} f(x) = L}_{\text{Hypothesis}} \implies \lim_{x \to c} |f(x)| = |L|$.

Hint: $\big|\,|f(x)| - |L|\,\big| \leq |f(x) - L|$ reverse Δ-ineq.

Hypothesis: Let $\varepsilon > 0$ be given ⊛ $\exists \delta > 0$ $\forall x \in D$
$(0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon)$

$\forall \varepsilon > 0$ $\exists \delta > 0$ as above in ⊛ $\forall x \in D$
$0 < |x - c| < \delta \implies \big|\,|f(x)| - |L|\,\big| \leq |f(x) - L| < \varepsilon$. #.

5.2 Continuous functions.

Defn $f : D \to \mathbb{R}$, $c \in D$.

(i) f is continuous at c if

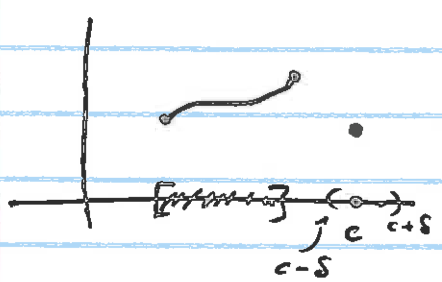
$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D\ \left(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \right)$$

(ii) f is continuous on D if it is continuous at all $c \in D$.

Remark: ① In def of limit we required $c \in D'$, but in def of continuity we require $c \in D$.

② if $c \in D$, but $c \notin D'$, i.e. c is an isolated pt

then f is automatically continuous at c.



$\left.\begin{array}{l} c \notin D' \\ c \in D \end{array}\right\}$ isolated pt of domain

$\exists\, \delta > 0$ s.t. $\left.\begin{array}{l} x \in D \\ |x - c| < \delta \end{array}\right\} \Rightarrow x = c$

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D\ \left(|x - c| < \delta \Rightarrow x = c \text{ and } |f(x) - f(c)| = 0 < \varepsilon . \right)$$

Obs Every function $f : \mathbb{N} \to \mathbb{R}$ is continuous

$f : \mathbb{Z} \to \mathbb{R}$ " " .

<u>Thm:</u> Let $f: D \to \mathbb{R}$, $c \in D \cap D'$.
Then the following are equivalent

(1) f is continuous at c ($\overset{\text{Defn.}}{\forall \varepsilon > 0 \, \exists \delta > 0 \cdots}$)

(2) $\forall$ sequence $(x_n) \subseteq D$, $x_n \to c \implies f(x_n) \to f(c)$

<u>Calc I</u> $\to$ (3) $\lim_{x \to c} f(x) = f(c)$

(4) $\forall$ neighbourhood U of $f(c)$ $\exists$ neighborhood V of c s.t. $f(V \cap D) \subseteq U$.

<u>Proof</u> ① $\iff$ ③ follows from defs of limits & continuity

③ $\iff$ ② Thm 5.1.8

How do we prove ① $\iff$ ④?

$|x-c| < \delta$
$-\delta < x-c < +\delta$
$c-\delta < x < c+\delta$
$x \in N(c,\delta)$

f continuous at c $\iff \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in D (|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

$\iff \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in D (x \in N(c,\delta) \implies f(x) \in N(f(c), \varepsilon))$

$\iff \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x$
$(x \in D \cap N(c,\delta) \implies f(x) \in N(f(c), \varepsilon)$

$\iff \forall \varepsilon > 0 \, \exists \delta > 0$ $f(D \cap \underbrace{N(c,\delta)}_{V}) \subseteq \underbrace{N(f(c), \varepsilon)}_{U}$

$\iff \forall$ neighborhood U of $f(c)$ $\exists$ neigborhood V of c s.t. $f(V \cap D) \subseteq U$.

Ex
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}-\mathbb{Q}. \end{cases}$$

f is <u>nowhere</u> continuous.

Let $c \in \mathbb{R}$

Density of $\mathbb{Q}$ in $\mathbb{R}$

$\forall n \in \mathbb{N}\ \exists r_n \in \mathbb{Q}$ s.t. $c - \frac{1}{n} < r_n < c + \frac{1}{n}$

$|c - r_n| < \frac{1}{n}$

$r_n \to c$

Density of irrationals

$\forall n \in \mathbb{N}\ \exists q_n \in \mathbb{R}-\mathbb{Q}$ s.t. $c - \frac{1}{n} < q_n < c + \frac{1}{n}$

$q_n \to c$.

$|c - q_n| < \frac{1}{n}$

$f(r_n) = 1$, $r_n \in \mathbb{Q}$.
$f(q_n) = 0$, $q_n \in \mathbb{R}-\mathbb{Q}$.

$$\lim_{n\to\infty} f(r_n) = 1 \neq \lim_{n\to\infty} f(q_n) = 0$$

One cannot have both true $\left\{ f(c) = \lim_{n\to\infty} f(r_n) \text{ and } f(c) = \lim_{n\to\infty} f(q_n) \right.$

f is not continuous at c,
by previous thm. (2)

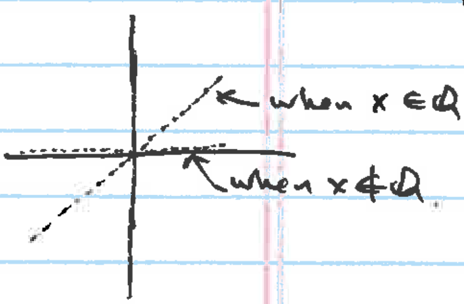
Ex b

$$g(x)=\begin{cases} x & \text{if } x\in\mathbb{Q}. \\ 0 & \text{if } x\in\mathbb{R}-\mathbb{Q}.\end{cases}$$

g is continuous only at 0.

$\forall \varepsilon>0 \ \exists \delta=\varepsilon>0$

$\forall x\in\mathbb{R} \ (\ |x-0|<\delta \Rightarrow |x|<\delta$, and



$|f(x)-0| = 0$ if $x\notin\mathbb{Q}$

$|f(x)-0| = |x|$ if $x\in\mathbb{Q}$.

$$|f(x)-0| \le \max(|x|,0) = |x| < \delta = \varepsilon.$$

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g is not continuous at $c\neq 0$.

As in previous example find

$r_n\in\mathbb{Q} \qquad r_n\to c$

$q_n\in\mathbb{R}-\mathbb{Q} \qquad q_n\to c$

$f(r_n)=r_n \longrightarrow c$

$\neq$

$f(q_n)=0 \longrightarrow 0$.

One cannot have both true

$$f(c)=\lim_{n\to\infty} f(r_n)=c \quad \text{and} \quad f(c)=\lim_{n\to\infty} f(q_n)=0$$

since $c\neq 0$.