5.1 Continue

Sequential characterization of limits of functions

Thm (5.1.8) Let $f : D \to \mathbb{R}$, $c \in D'$

$\lim_{x\to c} f(x) = L \iff$

$\forall$ sequence $(s_n) \subseteq D$, with $s_n \to c$, $s_n \neq c$ $\forall n$, one has $\lim_{n\to\infty} f(s_n) = L$

Proof

$(\Rightarrow:)$ Assume $\lim_{x\to c} f(x) = L$. ①

Given a sequence $(s_n) \subseteq D$, $s_n \xrightarrow{②} c$, $s_n \neq c$ $\forall n$

[Want to show $\lim_{n\to\infty} f(s_n) = L$

① $\overset{\text{defn.}}{\Longrightarrow}$ $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall x \in D$ $0 < |x - c| < \delta \overset{(*)}{\Rightarrow} |f(x) - L| < \varepsilon$

② $\Rightarrow$ Now given $\delta > 0$ $\exists N \in \mathbb{N}$ $\forall n \geq N$ $|s_n - c| < \delta$.

($\forall n$ $s_n \neq c$) so that

$0 < |s_n - c| < \delta \Rightarrow |f(s_n) - L| < \varepsilon$

by plugging $x = s_n$ into $(*)$

Combining all, we get:

Given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ $\forall n \geq N$, $|f(s_n) - L| < \varepsilon$.

i.e. $\lim_{n\to\infty} f(s_n) = L$.

For ($\Leftarrow$:) Use contrapositive

$\lim_{x \to c} f(x) \neq L \implies$ not ($\forall$ sequence $(s_n) \subseteq D$
with $s_n \to c$, $s_n \neq c$
one has $\lim_{n \to \infty} (f(s_n)) = L$)

To prove

$\exists$ sequence $(s_n) \subseteq D$, $s_n \to c$, $s_n \neq c$
and $\lim_{n \to \infty} f(s_n) \neq L$

Assume
not ($\lim_{x \to c} f(x) = L$)
not ($\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D \; (0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon)$)

Given $\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists x \in D \; (0 < |x - c| < \delta \;\&\; |f(x) - L| \geq \varepsilon)$

$\exists \varepsilon > 0 \; \forall \delta = \frac{1}{n}, n \in \mathbb{N} \; \exists x_n \in D$
$0 < |x_n - c| < \underset{\frac{1}{n}}{\delta_n}$ and $|f(x_n) - L| \geq \varepsilon$

$\exists \varepsilon > 0 \; \forall n \in \mathbb{N} \; \exists x_n \in D$
$\underbrace{0 < |x_n - c| < \frac{1}{n}}_{\substack{x_n \to c \\ x_n \neq c}}$ but $\underbrace{|f(x_n) - L| \geq \varepsilon}_{f(x_n) \not\to L.}$

Method for showing (a) $\lim_{x\to c} f(x) \neq L$

OR

(b) $\lim_{x\to c} f(x)$ DNE.

(a) Find a sequence $x_n \xrightarrow{\neq} c$ $f(x_n) \not\to L$.

(b) Find a sequence $x_n \xrightarrow{\neq} c$ $f(x_n)$ diverges

OR

Find two sequences $x_n \xrightarrow{\neq} c$ $f(x_n) \to L_1$

$y_n \xrightarrow{\neq} c$ $f(y_n) \to L_2$ $\neq$.

Ex(a) $\lim_{x\to 0} \sin\frac{1}{x}$ DNE

Recall $\sin\left(\frac{\pi}{2}+2k\pi\right) = 1$ $\forall k \in \mathbb{Z}$

$\sin\left(\frac{3\pi}{2}+2k\pi\right) = -1$. $\forall k \in \mathbb{Z}$.

Take $x_n = \dfrac{1}{\frac{\pi}{2}+2n\pi} \longrightarrow 0$ as $n \to \infty$.

Take $y_n = \dfrac{1}{\frac{3\pi}{2}+2\pi\cdot n} \longrightarrow 0$ as $n\to\infty$

$f(x) = \sin\frac{1}{x}$ $\quad f(x_n) = \sin \dfrac{1}{\frac{1}{\frac{\pi}{2}+2\pi n}} = \sin\left(\frac{\pi}{2}+2\pi n\right) = 1$

$f(y_n) = \qquad = -1$

$\lim_{n\to\infty} f(x_n) = 1$, $\lim_{n\to\infty} f(y_n) = -1$ $\overset{\text{Thm 5.1.8}}{\Longrightarrow}$ $\lim_{x\to 0} \sin\frac{1}{x}$ DNE

Ex(b) Prove $\lim_{x\to 0} x\sin\frac{1}{x} = 0$. from defn of limit.

We want $\forall \varepsilon>0\ \exists \delta>0\ \forall x\in D=\mathbb{R}-\{0\}$
$0<|x-0|<\delta \Rightarrow \left|x\sin\frac{1}{x}-0\right|<\varepsilon$.

Proof $\forall \varepsilon>0\quad \exists\ \delta=\varepsilon\quad \forall x\in\mathbb{R}-\{0\}$,

$$0<\underbrace{|x-0|}_{|x|}<\delta \Rightarrow \left|x\sin\frac{1}{x}-0\right| = |x|\underbrace{\left|\sin\frac{1}{x}\right|}_{\le 1} \le |x|<\delta=\varepsilon.$$

5.1.13 Thm: If $\left.\begin{array}{l} f: D\to\mathbb{R}, \\ g: D\to\mathbb{R},\end{array}\right\}$ $c\in D'$ and $\lim_{x\to c} f(x)=L$
$\lim_{x\to c} g(x)=M$.

then
$\lim_{x\to c}(f(x)+g(x))=L+M$
$\lim_{x\to c}(f(x)\cdot g(x))=LM$
$\lim_{x\to c}(kf(x))=kL$
$\lim_{x\to c} f(x)/g(x)=L/M$ provided that $M\neq 0$
$g(x)$ is not 0
in a neigh. of c.

How to read p201.

Proof: Idea
$f(x)\to L \quad g(x)\to M$ $\qquad (f+g)(x)\to L+M$

$\forall s_n \xrightarrow{\neq} c$: Thm 5.1.8 $\qquad$ 5.1.8 $\forall s_n\xrightarrow{\neq} c$

$f(s_n),\ g(s_n) \xrightarrow[\text{4.2.1 p171}]{\text{Thm}} f(s_n)+g(s_n)\to L+M$
$\downarrow \qquad \downarrow$
$L \qquad M$

Ex Exercise

p 203 5.1. # 2(a)

For every polynomial $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $a_i \in \mathbb{R}$.

one has $\lim_{x \to c} P(x) = P(c)$.

Proof by Induction:

Base cases (i) $p(x) = a_0$ constant polynomial

$\lim_{x \to c} a_0 = a_0$, because

$\forall \varepsilon > 0 \; \exists \delta > 0$ anything you want

$\forall x \in \mathbb{R} \quad 0 < |x - c| < \delta \Rightarrow |p(x) - a_0| = |a_0 - a_0| = 0 < \varepsilon.$

$\lim_{x \to c} x = c$, because

$\forall \varepsilon > 0 \; \exists \delta = \varepsilon$

$\forall x \in \mathbb{R} \quad 0 < |x - c| < \delta \Rightarrow |\overset{x}{\overset{\parallel}{p(x)}} - \overset{c}{\overset{\parallel}{p(c)}}| = |x - c| < \delta = \varepsilon.$

Inductive step (ii) $q(k) \Rightarrow q(k+1) \; \forall k \in \mathbb{N}$

$q(k)$: For every polynomial $P(x)$ of degree $\leq k$, we have $\lim_{x \to c} P(x) = P(c)$.

Let $Q(x)$ be a polynomial of degree $k+1$

$$Q(x) = \underbrace{a_0 + a_1x + \cdots + a_kx^k}_{P(x)} + a_{k+1}x^{k+1}$$

degree k or less.

$$\lim_{x \to c} Q(x) = \lim_{x \to c} \left(P(x) + a_{k+1} \cdot x^{k+1} \right)$$

by Thm 5.1.13 p201 & induction hypothesis

$$= \lim_{x \to c} P(x) + \lim_{x \to c} a_{k+1} x^{k+1}$$

$$= P(c) + a_{k+1} \lim x^{k+1}$$

$$= P(c) + a_{k+1} \lim_{x \to c} x^k \cdot x$$

$$= P(c) + a_{k+1} \underbrace{(\lim x^k)}_{c^k} \underbrace{(\lim x)}_{c}$$

$$= P(c) + a_{k+1} c^{k+1} = Q(c).$$

Hence by a proof by induction,

$\forall$ polynomial $Q(x)$ of degree n,

$$\lim_{x \to c} Q(x) = Q(c).$$