Oct 31, 2018

(*)(*)(*) **Thm** BOLZANO-WEIERSTRASS

Every bounded and infinite subset A of $\mathbb{R}$ must have an accumulation point in $\mathbb{R}$, that is $A' \neq \emptyset$.

Not in the book

Proof: (LUB prop $\Rightarrow$ BW)

Let A be an infinite & bounded subset of $\mathbb{R}$.

$\exists M \in \mathbb{R}$ s.t.

$$A \subseteq [-M, M]$$

Let $S = \{x \in \mathbb{R} \mid$ There are finitely many $a \in A$ (or none) s.t. $a < x\}$

$S \neq \emptyset$, since $-M \in S$.

S is bounded above by M

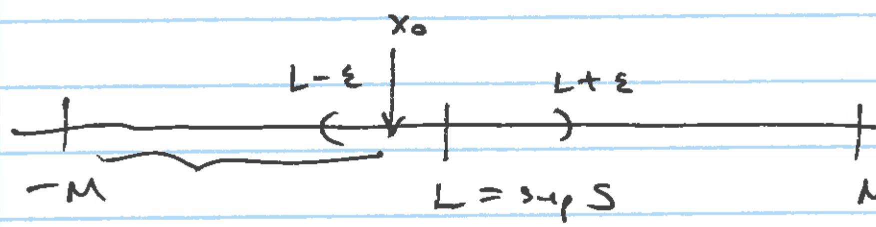
(since for any $x > M$, there are (all of A) infinitely many elements of A s.t. $a < x$.

Hence by LUB property (completeness Axiom)

$\sup S = L$ exists & $L \in \mathbb{R}$.

Claim $L \in A'$.

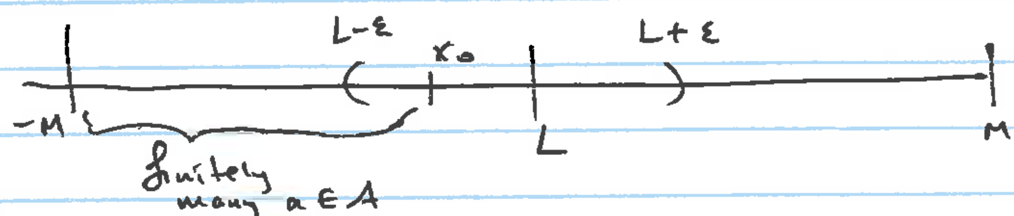
Recall $\underline{\text{defn}}$. $\forall \varepsilon > 0$ $\underbrace{N^*(L,\varepsilon)}_{(L-\varepsilon,\,L+\varepsilon)-\{L\}} \cap A \neq \phi$. Want to show.



$L = \sup S$ $\quad$ $L-\varepsilon$ is not an upper bd for S

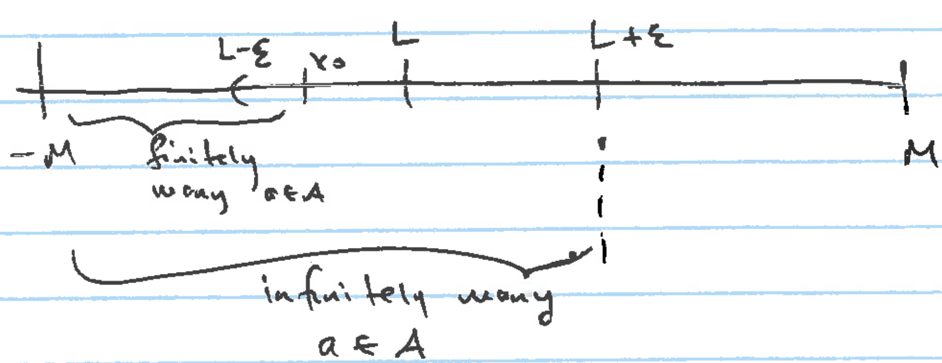
$\exists x_0 \in S$, $L-\varepsilon < x_0$

There are finitely many (or none) $a \in A$ s.t. $a < x_0$.



$\sup S = L < L+\varepsilon \notin S$

There are infinitely many $a \in A$ $\quad a < L+\varepsilon$.



$\Rightarrow$ There are infinitely many $a \in A$ $\quad x_0 \leq a < L+\varepsilon$.

" " " " $\quad a \in A$ $\quad L-\varepsilon < a < L+\varepsilon$

By removing L, in $(L-\varepsilon, L+\varepsilon)-\{L\}$ we still have $N^*(L,\varepsilon) \cap A$ with infinitely many elts.

$N^*(L,\varepsilon) \cap A \neq \phi$, $\forall \varepsilon$. $\qquad L \in A' \neq \phi$. #.

In $\mathbb{R}$: Compact $\Longleftrightarrow$ Closed & Bounded

Exs (All in $\mathbb{R}$)

$\emptyset$ is compact

$\mathbb{R}$ is not compact, not bounded (but closed)

$S = \{1, 2, 3, 4, 5, \ldots 100\}$ compact, bounded

$Bd\,S = S \subseteq S$.

$\forall a, b \in \mathbb{R}$, $[a,b]$ is compact

(a,b) is not compact

$(a,b]$ " not "

$[a,b)$ " not "

$[a,\infty)$ " not "

(a,∞) " not ".

All of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$: unbounded, not compact.

$S = \{\frac{1}{n} : n \in \mathbb{N}\}$ not closed, $0 \in S'$, $0 \notin S$

S is not compact

$T = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.

In $\mathbb{R}$: Lemma: if S is bounded, $\bar{S} = cl\,S$ is compact.

$U = [0,1) \cup (1,2]$ not compact, since $1 \in bd\,U$

$1 \notin U$

U is not closed.

Ex **

$S = \{ s_n \mid n \in \mathbb{N}$ defined as below$\}$.

s_n = truncation of $\sqrt{2}$ at the n^{th} decimal

$$\left.\begin{array}{l} s_1 = 1.4 = \frac{14}{10} \in \mathbb{Q} \\ s_2 = 1.41 = \frac{141}{100} \in \mathbb{Q} \\ s_3 = 1.414 = \frac{1414}{1000} \in \mathbb{Q} \\ \vdots \end{array}\right\} \quad \begin{array}{l} \forall n \ s_{n+1} \geq s_n \\ \forall n \quad s_n \leq 2 \end{array}$$

$s_n \to \sqrt{2}$ in $\mathbb{R}$.

$\lim_{n\to\infty} s_n$ DNE in $\mathbb{Q}$.

S is a bounded & infinite subset of $\mathbb{R}$.

BW $\Rightarrow$ S has an accumulation pt : $\sqrt{2}$

in $\mathbb{R}$

S is a bounded & infinite subset of $\mathbb{Q}$

S has no accumulation pt in $\mathbb{Q}$.

i.e. Bolz. Weier. is false in $\mathbb{Q}$.

True / but no proof : $[0, 2] \cap \mathbb{Q} = T$

if T is considered as a subset of $\mathbb{Q}$, (not $\mathbb{R}$)

then T is closed & bounded in $\mathbb{Q}$

but T is not compact. (in terms of open covers.)

if T is considered as a subset of $\mathbb{R}$

then T is bounded

T is not closed, $\overline{T} = [0, 2] \neq T$.

T is not compact.

Heine - Borel theorem fails in $\mathbb{Q}$.

(Heine - Borel theorem is true in $\mathbb{R}$.)

BIG PICTURE

Every bounded monotone sequence converges in $\mathbb{R}$

$\Longleftrightarrow$

Every Cauchy sequence converges in $\mathbb{R}$

Completeness: LUB property

Every non-empty subset of $\mathbb{R}$ bounded above has a supremum in $\mathbb{R}$

BW

Every bounded & infinite subset A of $\mathbb{R}$, has an accumulation pt in $\mathbb{R}$.

$\Longleftrightarrow$

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence, with limit in $\mathbb{R}$.

All are True in $\mathbb{R}$

All are False in $\mathbb{Q}$ (counterexample: see previous page ④ Ex **

$S = \{ s_n \mid n \in \mathbb{N} \}$

$\Uparrow$ $\mathbb{Q}$

s_n = truncation of $\sqrt{2}$ at n^{th} decimal

$\lim s_n$ DNE in $\mathbb{Q}$; $s_n \to \sqrt{2} \notin \mathbb{Q}$ in $\mathbb{R}$