Oct 26, 2018

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3.4 continue

Prop: S is open $\iff$ $\mathbb{R} - S$ is closed

Prop: (a) Intersection of any collection of closed sets is closed

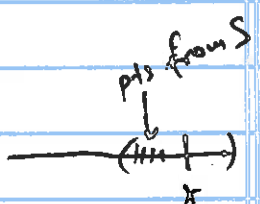
(b) Union of finitely many closed sets is closed

III

Defn Given a set $S \subseteq \mathbb{R}$.

A point $x \in \mathbb{R}$ is called an accumulation pt of S if $\forall \varepsilon > 0$ $\underbrace{N^*(x, \varepsilon)} \cap S \neq \emptyset$.

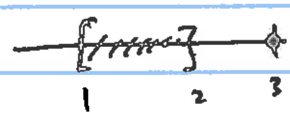
$N(x, \varepsilon) - \{x\} = (x - \varepsilon, x) \cup (x, x + \varepsilon)$.



The set of accumulation pts of S is denoted by S'

A pt $x \in S$ is called isolated if $x \notin S'$.

Ex



$S = [1, 2] \cup \{3\}$.

3 is isolated $\begin{cases} 3 \in S \\ 3 \notin S' = [1, 2] \end{cases}$

Lemma: (1) $x \in S' \iff \begin{cases} \exists \text{ sequence } (s_n) \text{ in } S \text{ s.t.} \\ s_n \neq x \ \forall n. \\ s_n \to x. \end{cases}$

Defn Given $S \subseteq \mathbb{R}$, One defines the closure

$cl(S) = \overline{S} = S \cup S'$

Books notation — standard notation.

Compare Lemma 1 & Lemma 2

Lemma 2 $x \in \overline{S} = \mathrm{cl}\, S \iff \exists$ sequence (s_n) in S s.t. $s_n \to x$.

$\overline{S} = S \cup S'$

Prop:

$\mathrm{bd}\, S \subseteq S \iff S$ is closed $\iff S' \subseteq S$

$\Updownarrow$

1. Learn! without proof.

$\mathrm{cl}\, S = \overline{S} = S$

2. $\mathrm{cl}\, S = \overline{S} = S \cup S' = S \cup \mathrm{bd}\, S = \mathrm{int}\, S \cup \mathrm{bd}\, S$

CRUCIAL Given $S \subseteq \mathbb{R}$.

$x \in \mathrm{int}\, S \implies x \in S'$ always!

$x \in \mathrm{ext}\, S \implies x \notin S'$ "

inconclusive

$x \in \mathrm{bd}\, S \rightarrow$ either $x \in S'$ or $x \notin S'$

$x \in S' \rightarrow$ either $x \in \mathrm{bd}\, S$ or $x \notin \mathrm{bd}\, S$

See examples next pages.

Examples

	interior	S open?	S closed?	bd	S'	$\overline{S}$
$S=(0,1]$	$(0,1)$	NO	NO	$\{0,1\}$	$[0,1]$	$[0,1]$.
		$S \neq \text{int} S$	$\text{bd} S \not\subseteq S$			
$S=[1,5]$	$(1,5)$	NO	YES	$\{1,5\}$	$[1,5]$	$[1,5]$.

$S=[0,3)\cup(3,4]$

$\text{int} S=(0,3)\cup(3,4)$ not closed $S'=[0,4]$.

not open $\text{bd}=\{0,3,4\}$. $\overline{S}=[0,4]$.

$S=(0,1)\cup\{6\}$

$\text{int} S=(0,1)\neq S$ $\text{bd} S=\{0,1,6\}\not\subseteq S$

S is not open. S is not closed

$\forall_{0<\varepsilon<5}$ $\underbrace{N^*}_{\text{6 is not in it.}}(6,\varepsilon)\cap S=\emptyset$. $*$ $S'=[0,1]$

$\overline{S}=[0,1]\cup\{6\}$

$6\in \text{bd} S$, $6\notin S'$

$\frac{1}{2}\notin \text{bd} S$ $\frac{1}{2}\in S'$

$\emptyset$ $\text{int}\emptyset=\emptyset$ open closed $\text{bd}\emptyset=\emptyset$ $S'=\emptyset$ $\overline{S}=\emptyset$.

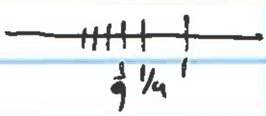
$\mathbb{R}$ $\text{int}\mathbb{R}=\mathbb{R}$ open closed $\text{bd}\mathbb{R}=\emptyset$ $\mathbb{R}'=\mathbb{R}$ $\overline{\mathbb{R}}=\mathbb{R}$.

More examples

	Interior	open?	closed?	bd	S'	$\overline{S}$
$S =$ finite set (non-empty) $\cancel{\phi}$	ϕ	no	yes; $\{x\}$ closed; union of finitely many closed sets is closed.	$bd\, S = S$	ϕ	$\overline{S} = S$.
$\mathbb{Z}$ all integers	ϕ	no	yes	$bd\, \mathbb{Z} = \mathbb{Z}$	ϕ; all pts of $\mathbb{Z}$ are isolated	$\overline{\mathbb{Z}} = \mathbb{Z}$

all positive integers $\mathbb{N}$ behaves same.

$S = \{ \frac{1}{n^2} \mid n \in \mathbb{N} \}$



	Interior	open?	closed?	bd	S'	$\overline{S}$
$S = \{ \frac{1}{n^2} \mid n \in \mathbb{N} \}$	$Int\, S = \phi$	no	no; $\frac{1}{n^2} \to 0 \notin S$, $0 \notin S'$	$bd\, S = S \cup \{0\} = \overline{S}$	$S' = \{0\}$	

$S = \mathbb{Q}$

$Int\, \mathbb{Q} = \phi$ — every non-empty open interval contains irrationals

$\left[\begin{array}{l} Int\, \mathbb{Q} \neq \mathbb{Q} \\ \text{not open} \end{array} \right.$

$\left[\begin{array}{l} \text{not closed} \\ bd\, \mathbb{Q} \not\subseteq \mathbb{Q} \end{array} \right.$

$bd\, \mathbb{Q} = \mathbb{R}$.

$S' =$ the set of all possible limits of rational sequences ($\neq$ limit)

$\mathbb{Q}' = S' = \mathbb{R}$.

$\overline{\mathbb{Q}} = \mathbb{R}$.

Same for $Ir = \mathbb{R} \setminus \mathbb{Q}$.