Oct 24, 2018

Reminders

Q4 tomorrow 10/25

MT2 11/2/18

Beginning - 3.4 included

4.1-3 included

Review Session Oct 31 Wed 7:30-9:00 pm

Continue 3.4

To prove

(3.4.7) Thm (ii) Let $S \subseteq \mathbb{R}$

S is open $\Longleftrightarrow$ $\mathbb{R}-S$ is closed.

Proof Let $T = \mathbb{R} \setminus S$

$bd(S) = bd(\mathbb{R} \setminus S) = bd(T)$, since

$\forall x \in bd(S)$ $\forall \varepsilon > 0$ $N(x,\varepsilon) \cap S \neq \emptyset$

$N(x,\varepsilon) \cap (\mathbb{R}-S) \neq \emptyset$

$\forall x \in bd(\mathbb{R}-S)$ $\forall \varepsilon > 0$ $N(x,\varepsilon) \cap (\mathbb{R}-S) \neq \emptyset$

$N(x,\varepsilon) \cap (\underbrace{\mathbb{R}-(\mathbb{R}-S)}_{S = \mathbb{R}-T}) \neq \emptyset$.

$int\ S = ext(\mathbb{R}-S) = ext\ T$

$int\ T = int(\mathbb{R}-S) = ext(S)$

$$S \text{ is open} \overset{\text{defn}}{\iff} \operatorname{bd} S \cap S = \emptyset$$

$$\iff \operatorname{bd}(\mathbb{R}-S) \cap S = \emptyset$$

$$\iff \operatorname{bd}(\mathbb{R}-S) \subseteq \mathbb{R}-S$$

$$\overset{\text{defn.}}{\iff} \mathbb{R}-S \text{ is closed} \qquad \#$$

<u>Prop</u> $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$ is an open set.

<u>Recall Thm(?)</u> S is open $\iff S = \operatorname{int} S$,
which means that if you want to show a set is open, it suffices to show all pts of S are interior pts of S.

Proof

Let $x \in (a,b)$ be given.

Let $\varepsilon = \min(x-a,\, b-x) > 0$.

$\forall y \in N(x,\varepsilon)$

$$|y-x| < \varepsilon$$

$$-\varepsilon < y-x < \varepsilon$$

$$\varepsilon \le b-x$$

$$\varepsilon \le x-a$$

$$a \le x-\varepsilon < y < x+\varepsilon \le b$$

$$a < y < b \quad , \quad y \in (a,b)$$

$\forall y \in N(x,\varepsilon)$, $y \in (a,b)$

$N(x,\varepsilon) \subseteq (a,b)$ $\quad x$ is an interior pt. of (a,b).

For all subsets of $\mathbb{R}$:

Prop (a) Union of any collection of open sets is open.

(b) Intersection of finitely many open sets is open.

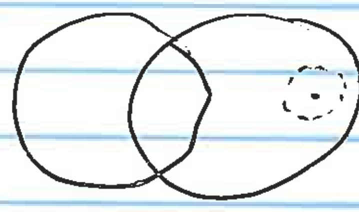
Ex

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

$\forall n \in \mathbb{N}$ $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is an open set

$\{0\}$ is not an open set.

Proof of Prop



(a) U_α, $\alpha \in \Lambda$ be a collection of open sets

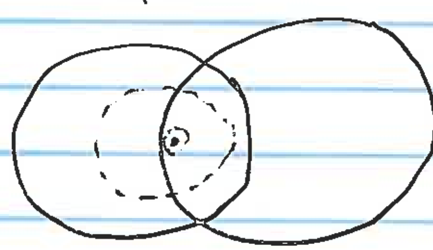
$p \in \bigcup_{\alpha\in\Lambda} U_\alpha = W$ then $p \in U_{\alpha_0}$ for some α_0

U_{α_0} is open.

$\exists\, \varepsilon > 0$ $N(p, \varepsilon) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

every pt of W is an interior pt of W.

(b)



Let $U_1, U_2 \ldots U_\ell$ be a finite collection of open sets.

$$V = \bigcap_{n=1}^{\ell} U_n \quad , \text{ let } p \in V.$$

each open

Then $p \in U_1$, $p \in U_2$, ... and, $p \in U_\ell$

$\exists\, \varepsilon_1 > 0 \quad N(p, \varepsilon_1) \subseteq U_1$, and
$\exists\, \varepsilon_2 > 0 \quad N(p, \varepsilon_2) \subseteq U_2$, and
$\vdots$, and
$\exists\, \varepsilon_\ell > 0 \quad N(p, \varepsilon_\ell) \subseteq U_\ell$

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \dots \varepsilon_\ell) > 0$

then $\varepsilon \leq \varepsilon_n$

$\forall n = 1, \dots \ell \quad N(p, \varepsilon) \subseteq U_n$

$$N(p, \varepsilon) \subseteq \bigcap_{n=1}^{\ell} U_n$$

Hence $\forall p \in V$, p is an interior pt of V.

(Exc. 26 p 143) **Thm:** Let U be a subset of $\mathbb{R}$.

You may assume without proof.

U is open $\iff$ U is a union of countably many disjoint open intervals.

****

Exs

(i) $(3,5) \cup (7,8) \cup \underbrace{(8, \infty)}$ open. by previous Thm.

$$(8, \infty) = \bigcup_{n=9}^{\infty} (8, n)$$

(ii) Is $\{x\}$ closed? $\mathbb{R} - \{x\} = \underbrace{(-\infty, x) \cup (x, \infty)}_{\text{open.}}$

$\mathbb{R} - \{x\}$ is open $\implies$ $\{x\}$ is closed.

(iii) $S = [1,2] \cup \{7\} \cup [9, \infty)$

open? No, since $7 \notin \operatorname{int} S$.

closed? YES, since

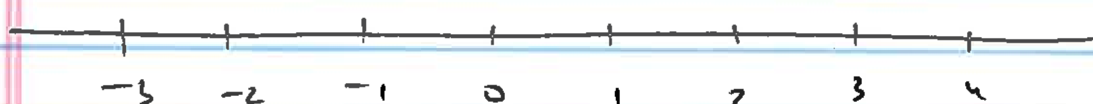
$$\mathbb{R} - S = \underbrace{(-\infty, 1) \cup (2,7) \cup (7,9)}_{\text{open}}$$

(iv) $\mathbb{Z} = \{ n \mid n \text{ whole number} \}$

$= \{0, 1, -1, 2, -2, 3, -3, \ldots \}$

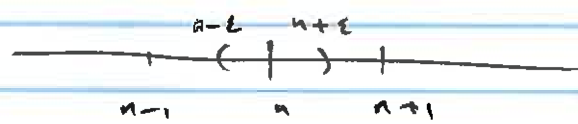
$\mathbb{Z}$ is closed since

$\mathbb{R} - \mathbb{Z} = \bigcup_{n=-\infty}^{\infty} (n, n+1)$ open.



Is $\mathbb{Z}$ open? $\quad \operatorname{int} \mathbb{Z} = \phi$

$\forall \varepsilon > 0 \ \forall n \in \mathbb{Z}, \ N(n, \varepsilon) \cap (\mathbb{R} - \mathbb{Z}) \neq \phi$.



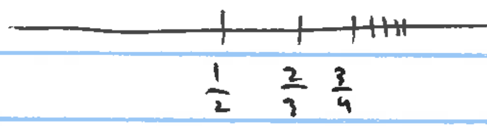
$\phi = \operatorname{int} \mathbb{Z} \neq \mathbb{Z}. \implies \mathbb{Z}$ is not open.

Ex. (v)

$$S = \left\{ \frac{n}{n+1} \;\middle|\; n \in \mathbb{N} \right\}$$

$\frac{n}{n+1} \to 1$ as $n \to \infty$.

Look at example done on 10/22/18 page 2



$\text{int } S = \emptyset$

$\text{bd } S = S \cup \{1\}$

S open ~~YES~~ or NO ? $\quad \text{int } S = \emptyset \neq S$.

S closed ~~YES~~ or NO ? $\quad S$ closed $\iff \text{bd } S \subseteq S$

$\underbrace{S \text{ closed}}_{\text{bd} S \not\subseteq S}$ $\qquad \text{bd } S$

(vi) $\mathbb{Q} \subseteq \mathbb{R}$

$\text{int } \mathbb{Q} = \emptyset$

$\text{bd } \mathbb{Q} = \mathbb{R}$

$\mathbb{Q}$ is not open $\qquad \text{int } \mathbb{Q} = \emptyset \neq \mathbb{Q}$.

$\mathbb{Q}$ is not closed $\qquad \mathbb{R} = \text{bd } \mathbb{Q} \not\subseteq \mathbb{Q}$

Every non-empty open interval contains infinitely many rationals
" " irrationals
by Density of rationals & Density of irrationals

Similarly $\underbrace{Ir}_{\text{all Irrational numbers}} = \mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$

$\text{int}(Ir) = \emptyset$

$\text{bd}(Ir) = \mathbb{R}$

Ir is not open

Ir is not closed