3.4

Ex $S = (-\infty, 1) \cup [2,3) \cup (3,4) \cup \{5\}$

without proof.

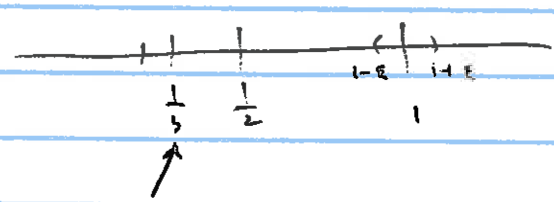
$$\begin{cases} bd\, S = \{1, 2, 3, 4, 5\} \\ int\, S = (-\infty, 1) \cup (3,4) \cup (2,3) \\ ext\, S = int(\mathbb{R} - S) = (1,2) \cup (4,5) \cup (5, \infty) \end{cases}$$

Exc. 3.4: 3 & 4 a,c p141

(a) $S = \{\frac{1}{n} \mid n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Claim $int\, S = \emptyset$.

$x \in int\, S \overset{\text{defn}}{\iff} \exists \varepsilon > 0 \quad N_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \subseteq S$.



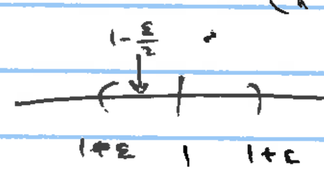
Recall $int\, S \subseteq S$.

$x \notin int\, S \iff \forall \varepsilon > 0 \quad (x-\varepsilon, x+\varepsilon) \not\subseteq S$

$\Updownarrow$

$(x-\varepsilon, x+\varepsilon) \cap (\mathbb{R} - S) \neq \emptyset$

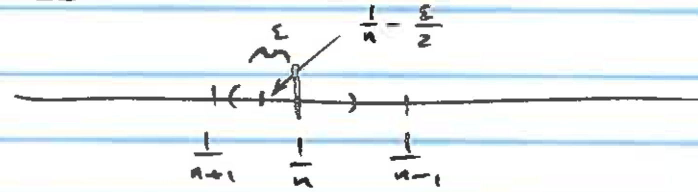
$1 \notin int\, S$, since $\forall \varepsilon > 0$



$1 - \frac{\varepsilon}{2} \in (1-\varepsilon, 1+\varepsilon)$ & $1 - \frac{\varepsilon}{2} \notin S$ if $\varepsilon \leq \frac{1}{2}$

$\frac{3}{4} \in (1-\varepsilon, 1+\varepsilon)$ & $\frac{3}{4} \notin S$. if $\varepsilon > \frac{1}{2}$

$\frac{1}{n} \notin \operatorname{int} S$ since



$\forall \varepsilon > 0$ $\quad \left.\begin{array}{l} \frac{1}{n} - \frac{\varepsilon}{2} \notin S \\ \frac{1}{n} - \frac{\varepsilon}{2} \in \left(\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right) \end{array}\right\}$ if $\varepsilon < \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$\underbrace{\frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right)}_{\text{midpt of } \frac{1}{n},\, \frac{1}{n+1}} \notin S$ $\quad$ if $\quad \varepsilon \geq \frac{1}{n(n+1)}$ ← when

; also $\frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right) \in \left(\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right)$

$\Rightarrow$ This proves $\operatorname{int} S$ contains no pts of S. $\operatorname{int} S \subseteq S$. $\operatorname{int} S = \emptyset$.

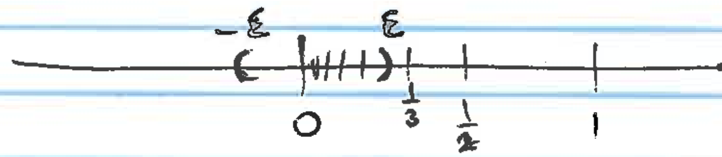
To show $\underline{\operatorname{bd} S}$ $\quad \left\{\frac{1}{n} \;\middle|\; n \in \mathbb{N}\right\} \cup \{0\} \subseteq \operatorname{bd} S$

Given $n \in \mathbb{N}$; $\forall \varepsilon > 0$ $\quad$ above shows $N\left(\frac{1}{n}, \varepsilon\right) \cap (\mathbb{R} - S) \neq \emptyset$

$\frac{1}{n} \in N\left(\frac{1}{n}, \varepsilon\right) \cap S \neq \emptyset$

$\forall n$, $\frac{1}{n} \in \operatorname{bd} S$. $\quad S \subseteq \operatorname{bd} S$

Claim:
$0 \in \operatorname{bd}(S)$ $\quad \forall \varepsilon > 0 \quad N(0, \varepsilon) = (-\varepsilon, \varepsilon)$



$-\frac{\varepsilon}{5} \in N(0, \varepsilon) \cap (\mathbb{R} - S) \neq \emptyset$

$\frac{1}{n} \in N(0, \varepsilon) \cap S \neq \emptyset$ $\quad$ if $\quad n > \frac{1}{\varepsilon}$, $n \in \mathbb{N}$
$\left(0 < \frac{1}{n} < \varepsilon\right)$

We established $\quad \left\{\frac{1}{n} \;\middle|\; n \in \mathbb{N}\right\} \cup \{0\} \subseteq \operatorname{bd} S$.
(Actually $=$)
due to time, we skip the details.

Exc 3.4. 5 & 4 p 141

(c)

$$T = \{ r \in \mathbb{Q} \mid 0 < r < \sqrt{2} \} \subseteq \mathbb{Q}$$

Claim int $T = \emptyset$.

$\forall \varepsilon > 0 \ \forall x \in T$ $\quad \overbrace{(x-\varepsilon, x+\varepsilon)}^{\text{interval}}$ contains $\quad x - \varepsilon < x + \varepsilon$

Density of rationals & irrationals $\Big\}$ infinitely many rational numbers, and infinitely many irrational numbers.

$$\text{int } T = \emptyset \Longleftarrow \emptyset \neq (x-\varepsilon, x+\varepsilon) \cap (\underbrace{\mathbb{R} - \mathbb{Q}}_{\text{irrationals}}) \subseteq (x-\varepsilon, x+\varepsilon) \cap (\mathbb{R} - T)$$

Also:

$$\left.\begin{array}{l} (x-\varepsilon, x+\varepsilon) \cap (\mathbb{Q}) \neq \emptyset \\ (x-\varepsilon, x+\varepsilon) \cap (T) \neq \emptyset \\ (x-\varepsilon, x+\varepsilon) \cap (\mathbb{R} - T) \neq \emptyset \end{array}\right\} \text{ If } x \in T, \forall \varepsilon > 0.$$

Claim' $T \subseteq \text{bd } T = [0, \sqrt{2}] = \{ y \in \mathbb{R} \mid 0 \leq y \leq \sqrt{2} \}$

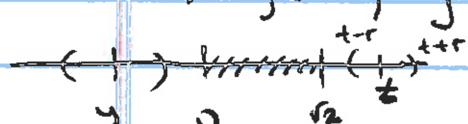
For bd $T \supseteq [0, \sqrt{2}]$. $\Bigg\{$ $\forall y \in [0, \sqrt{2}]$ $\forall \varepsilon > 0$

$(y-\varepsilon, y+\varepsilon)$ will contain rational pts from T

$(y-\varepsilon, y+\varepsilon)$ " " irrational " " $\mathbb{R} - T$

Notes Added: How do we know bd $T = [0, \sqrt{2}]$?

$\forall \ y < 0$, $y \in \text{ext}(T) = \text{int}(\mathbb{R} - T)$ since



$(+\frac{3}{2}y, \frac{1}{2}y) = N(y, \frac{|y|}{2}) \subseteq \mathbb{R} - T$.

$\forall t > \sqrt{2}$, for $r = \frac{t - \sqrt{2}}{2}$, $N(t, r) = (t-r, t+r) \subseteq (\sqrt{2}, \infty)$

since $t - r = \frac{t + \sqrt{2}}{2} > \sqrt{2}$.

$\Longrightarrow$ $\text{ext}(T) \supseteq (-\infty, 0) \cup (\sqrt{2}, \infty)$. bd $T \cap \text{ext } T = \emptyset \Longrightarrow \text{bd } T = [0, \sqrt{2}]$

OPEN & CLOSED SETS

Defn Let $S \subseteq \mathbb{R}$

S is called open if $\quad bd\, S \cap S = \emptyset$.

S is called closed if $\quad bd\, S \subseteq S$.

Caution A set can be both open & closed

A set can be neither open nor closed.

	open	closed
$(0,1)$	yes	no
$[0,1]$	no	yes
$(0,1]$	no	no
$\emptyset, \mathbb{R}$	yes	yes

$bd\, \emptyset = \emptyset$

$bd\, \mathbb{R} = \emptyset$

Thm: Let $S \subseteq \mathbb{R}$.

(i) S is open $\iff$ $S = int\, S$

(ii) S is open $\iff$ $\mathbb{R} - S$ is closed

Proof (i) ($\Rightarrow$:)

Assume S is open.

$bd\, S \cap S = \emptyset$. (defn) $\quad (*)$

$int\, S \subseteq S \quad$ (Proved it)

$S \cap ext\, S \subseteq S \cap (\mathbb{R} - S) = \emptyset$.

$(\mathbb{R} = int\, S \cup bd\, S \cup ext\, S) \Rightarrow \quad S \subseteq int\, S \cup bd\, S \quad **$

$*, ** \Rightarrow \quad S \subseteq int\, S$

$\longrightarrow S = int\, S$

$\Leftarrow$: Assume $S = \text{int } S$ (to prove S is open)

$\text{int } S \cap \text{bd } S = \emptyset$ proved it

$\text{bd } S \cap S = \emptyset$

S is open #.

(ii) To be done on Wednesday.