Overview of 3.3

Real Numbers is an ordered field, containing $\mathbb{Q}$ and satisfying the Completeness Axiom (the Least upper bound property.)

Thm: There exists a unique $\mathbb{R}$.

In $\mathbb{R}$ w/LUB property/completeness Axiom, All True:

- Archimedean Property
- Density of Rationals
- Density of Irrationals
- Every bounded monotone sequence converges.
- Every Cauchy sequence converges.

In $\mathbb{Q}$ the following are false

- Bounded monotone sequences converge? No
- Cauchy sequences converge? NO

In $\mathbb{R}$ Cauchy $\Rightarrow$ convergent (False in $\mathbb{Q}$)

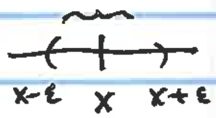
In $\mathbb{Q}$ & $\mathbb{R}$ convergent $\Rightarrow$ Cauchy

Ex $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} \in \mathbb{Q}$

(s_n) is Cauchy in $\mathbb{Q}$.

(s_n) is Cauchy in $\mathbb{R}$. $\longrightarrow$ $\lim s_n = + \ln 2 \notin \mathbb{Q}$.

3.4 Open & closed Sets / Topology of $\mathbb{R}$.



Defn (i) For $x \in \mathbb{R}$, $\varepsilon > 0$, A neighbourhood of x is a set of the form

$$N_\varepsilon(x) = N(x,\varepsilon) = \{ y \in \mathbb{R} \mid |y-x| < \varepsilon \}$$
$$= (x-\varepsilon, x+\varepsilon)$$

ε = radius of the open disc $N(x,\varepsilon)$

(ii) $N^*(x,\varepsilon) = N(x,\varepsilon) - \{x\} = (x-\varepsilon, x) \cup (x, x+\varepsilon)$ is called the deleted neighborhood of x. (of radius ε).

Defn (i) Given a set $S \subseteq \mathbb{R}$, $x \in S$.
x is called an interior pt of S if
$\exists \varepsilon > 0 \quad N(x,\varepsilon) \subseteq S$.

(ii) Given a set S, a point $x \in \mathbb{R}$ is called a boundary pt of S if
$\forall \varepsilon > 0 \quad \left(N(x,\varepsilon) \cap S \neq \emptyset, \text{ and } N(x,\varepsilon) \cap (\mathbb{R} - S) \neq \emptyset. \right)$

(iii) A point $x \in \mathbb{R}$ is called in the exterior of S (interior of the complement of S) if $\exists \varepsilon > 0 \quad N(x,\varepsilon) \cap S = \emptyset$.

Notation
- $\mathrm{int}(S)$ the set of all interior pts of S
- $\mathrm{bd}(S)$ the set of all boundary pts of S
- $\mathrm{ext}(S) = \mathrm{int}(\mathbb{R} - S)$

<u>Prop</u> (a) $\operatorname{int} S \subseteq S$

$\operatorname{ext} S \subseteq \mathbb{R} - S$

(b) $\mathbb{R} = \operatorname{int}(S) \cup \operatorname{ext}(S) \cup \operatorname{bd}(S)$

$$\left.\begin{array}{l} \operatorname{int}(S) \cap \operatorname{bd} S = \emptyset \\ \operatorname{int}(S) \cap \operatorname{ext} S = \emptyset \\ \operatorname{bd}(S) \cap \operatorname{ext} S = \emptyset. \end{array}\right\} \text{a partition}$$

<u>Proof</u> (a) Let $x \in \operatorname{Int} S$

x is an interior pt of S

$\exists \varepsilon > 0 \quad N(x, \varepsilon) \subseteq S$

$x \in (x - \varepsilon, x + \varepsilon) \subseteq S$

$\operatorname{int} S \subseteq S.$

$\operatorname{ext} S = \operatorname{int}(\mathbb{R} - S) \subseteq \mathbb{R} - S$

(b) $\underbrace{\operatorname{Int} S}_{\subseteq S} \cap \underbrace{\operatorname{ext} S}_{\subseteq \mathbb{R} - S} \subseteq S \cap (\mathbb{R} - S) = \emptyset.$

$x \in \operatorname{bd}(S) \iff \forall \varepsilon > 0 \ (N(x, \varepsilon) \cap S \neq \emptyset$ and $N(x, \varepsilon) \cap (\mathbb{R} - S) \neq \emptyset$

$x \notin \operatorname{bd}(S) \iff \exists \varepsilon > 0 \ (N(x, \varepsilon) \cap S = \emptyset$ OR $N(x, \varepsilon) \cap (\mathbb{R} - S) = \emptyset)$

$\iff \exists \varepsilon > 0 \ (N(x, \varepsilon) \subseteq \mathbb{R} - S$ OR $N(x, \varepsilon) \subseteq S)$

PTO ← $\circledast$ $\iff \left(\exists \varepsilon > 0 \ N(x, \varepsilon) \subseteq \mathbb{R} - S \right)$ OR $\left(\exists \varepsilon > 0 \ N(x, \varepsilon) \subseteq S \right)$

$\iff (x \in \operatorname{ext} S)$ OR $(x \in \operatorname{int} S).$

(*) Discussion

Can you distribute quantifier over $\vee$, $\wedge$?

True $\forall x \in \mathbb{N}$ (x is odd OR x is even)

False ($\forall x \in \mathbb{N}$ x is odd) OR ($\forall x \in \mathbb{N}$ x is even)

Allowed:

$$\forall x \; (p(x) \wedge q(x)) \equiv (\forall x \; p(x)) \wedge (\forall x \; q(x))$$

$$\exists x \; (p(x) \vee q(x)) \equiv (\exists x \; p(x)) \vee (\exists x \; q(x))$$

$$(\forall x \; p(x)) \vee (\forall q(x)) \Longrightarrow \forall x \; p(x) \vee q(x)$$

$$\not\Longleftarrow$$

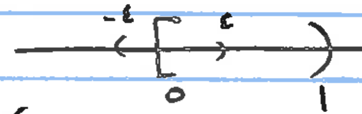
$$\exists x \; (p(x) \wedge q(x)) \Longrightarrow \exists x \, p(x) \wedge \exists x \; q(x)$$

$\not\Longleftarrow$ (may be getting different x's for $p(x)$ & $q(x)$)

Ex $S = [0,1) \subseteq \mathbb{R}$

$\text{bd } S = \{0, 1\}$

$0 \in \text{bd } S$ since



$\forall \varepsilon > 0 \quad N(0,\varepsilon) = (-\varepsilon, \varepsilon)$

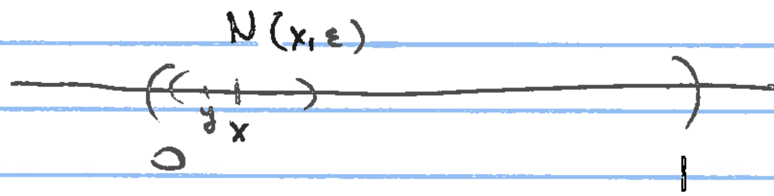
$(-\varepsilon, \varepsilon) \cap (\mathbb{R} - [0,1)) \neq \emptyset$ since $-\frac{\varepsilon}{2} \notin S$

$(-\varepsilon, \varepsilon) \cap [0,1) \neq \emptyset$ since $\frac{\varepsilon}{2} \in S \cap (-\varepsilon, \varepsilon)$ when $\varepsilon \leq 1$

$\frac{1}{2}, 0 \in S \cap (-\varepsilon, \varepsilon)$ when $\varepsilon > 1$.

$1 \in \text{bd } S$ (You do this)

Claim $\text{int } S = (0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$



Let $x \in (0,1)$ be given

take $\varepsilon = \min(x, 1-x) > 0$

To show $N(x,\varepsilon) = \{y \mid |y-x| < \varepsilon\} \subseteq [0,1) = S$

$\forall y \in N(x,\varepsilon) \quad |y-x| < \varepsilon$

$-\varepsilon < y - x < \varepsilon$

$0 \leq x - \varepsilon < y < x + \varepsilon \leq x + (1-x) = 1$

$\varepsilon \leq x \qquad \varepsilon \leq 1-x$

$0 < y < 1 \qquad y \in S$

PTO

We showed that:

$$\forall x \in (0,1)\ \exists\ \varepsilon = \underbrace{\min(x, 1-x)}_{>0}\ \underbrace{\forall y \in N(x,\varepsilon),\ y \in S}_{N(x,\varepsilon) \subseteq S}.$$

$$\forall x \in (0,1)\ \exists\ \varepsilon > 0\ \ N(x,\varepsilon) \subseteq S$$

$\forall x \in (0,1)$, x is an interior pt of S

$$(0,1) \subseteq \operatorname{int} S.$$

$$\operatorname{int} S \subseteq S = [0,1)$$

Since 0 is the only pt in S, which is not in $(0,1)$, and $0 \in \operatorname{bd} S$, $\operatorname{bd} S \cap \operatorname{int} S = \emptyset$, we conclude that

$$(0,1) = \operatorname{int} S.$$