Density of Rationals

Thm: $\forall x, y \in \mathbb{R} \; (x < y \Rightarrow \exists r \in \mathbb{Q} \;\; x < r < y)$.

Proof Case 1 $0 < x < y$ is done see notes 10/15/18

Case 2 $0 \leq x < y$

if $0 < x < y$, Done

if $0 = x < y$

$0 = x < \frac{x+y}{2} < y$

$\exists r \in \mathbb{Q}$

$x < \frac{x+y}{2} < r < y$

$\frac{x+y}{2} = \frac{y}{2} > 0$

Case 3 $x < 0 < y$, $0 \in \mathbb{Q}$.

Case 4 $x < y < 0$

$-x > -y > 0$

$\exists r \in \mathbb{Q}$.

$-x > r > -y$

$x < -r < -y$; $-r \in \mathbb{Q}$

rest are quite similar.

Thm: Density of irrationals in $\mathbb{R}$.

$\forall x, y \in \mathbb{R}$, $x < y$ $\exists$ a irrational, $a \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$x < a < y$$

Proof Take $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$

$$\Rightarrow \exists r \in \mathbb{Q} \text{ s.t. } \frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

$$x < \sqrt{2} r < y$$

if $r \neq 0$, we're done $\sqrt{2} r \notin \mathbb{Q}$

if $r = 0$, $\exists r' \in \mathbb{Q}$

$$\frac{x}{\sqrt{2}} < r = 0 < r' < \frac{y}{\sqrt{2}}$$

$r' \neq 0$

$$x < r'\sqrt{2} < y$$

$r'\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

p133 Exercise 10(a) $\forall x, y \in \mathbb{R}$, $x < y$ There are infinitely many $\{r_n \in \mathbb{Q}\}_{n \in \mathbb{N}}$ s.t. $\forall n \in \mathbb{N}$, $x < r_n < y$.

Sketch idea:

Find $r_1 \in \mathbb{Q}$. $x < r_1 < y$

Find $r_2 \in \mathbb{Q}$ $x < r_1 < r_2 < y$ $r_2 \neq r_1$

⋮

$\forall n \in \mathbb{N}$, given $x < r_1 < r_2 < \cdots < r_n < y$

Find $r_{n+1} \in \mathbb{Q}$. $x < r_1 < r_2 < \cdots < r_n < r_{n+1} < y$.

Exc 7a p 133

Ex $S_1 = \{1, 3, -6\}$, $2S_1 = \{2, 6, -12\}$

$S_2 = [2, 5)$

$-3S_2 = (-15, -6]$.

7a) [Given S is a bounded set
If $k \geq 0$ $\sup(kS) = k \sup S$

part of

If $k = 0$:

$$\sup\{0\} = \sup(0 \cdot S) = 0 \cdot \underbrace{\sup S}_{\in \mathbb{R}} = 0$$

S bdd

If $k > 0$

$\forall x \in S$ $x \leq \sup S$ (defn of sup(i))

$\forall x \in S$ $kx \leq k \sup S$

Every element of kS is of the form kx for some $x \in S$

$\forall y \in kS$, $y = kx \leq k \sup S$

kS is bounded above and $k \sup S$ is an upper bd for kS

$\circledast$ $\sup(kS) \leq k \sup S$ (def sup (ii))

Completeness Axiom $\exists$ least upper bd. an upper bd

Next we Need $\sup(kS) \geq k \sup S$.

we proved ⊛ $\forall k \in \mathbb{R}, k>0$, S bdd $\sup(kS) \le k \sup(S)$

substitute $\begin{cases} S \to kS \\ k \to \frac{1}{k} \end{cases}$

$$⊛ \implies \sup\Big(\underbrace{\tfrac{1}{k} kS}_{S}\Big) \le \frac{1}{k} \sup kS$$

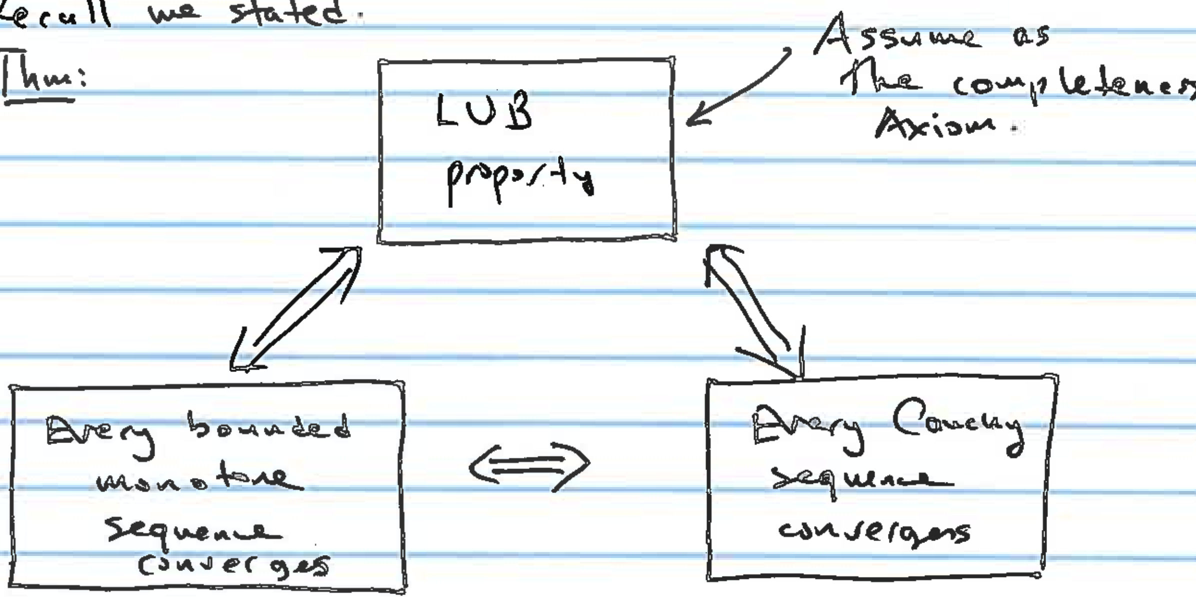
$$\sup(S) \le \frac{1}{k} \sup kS$$

$$⊛⊛ \quad k\sup(S) \le \sup kS$$

$$⊛ \ \& \ ⊛⊛ \implies k\sup(S) = \sup(kS)$$

Recall we stated.

Thm:



We will prove Thms I, II

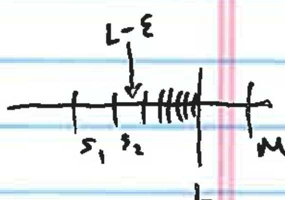
⊛⊛⊛ Thm: I LUB Property $\implies$ Every bounded monotone sequence converges.

Proof

Let (s_n) be an increasing bounded sequence. (decreasing case, HW)

s_n is bounded, $\exists M \in \mathbb{R}$ $\forall n \in \mathbb{N}$ $|s_n| \le M$.

Let $A = \{s_n \mid n \in \mathbb{N}\}$, set of values of (s_n).



A is bounded above. LUB Prop $\implies \exists \sup A = L \in \mathbb{R}$

$\forall \varepsilon > 0$ $L - \varepsilon$ is not an upper bd of A (Def of sup.)

$\exists s_N$ s.t $L - \varepsilon < s_N$, $\forall n \ge N$ $L - \varepsilon < s_N \le s_n \le L$.

$L - \varepsilon < s_n \le L \implies |s_n - L| < \varepsilon$. $\lim s_n = L$. #

(*)(*)(*) **Thm 2** LUB Prop $\Longrightarrow$ Every Cauchy sequence converges.

This proof is Not in the book. The proof in the book uses Bolzano - Weierstrass.

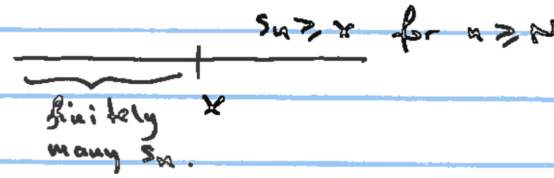
Proof: Let (s_n) be a Cauchy sequence.

(s_n) is a bounded sequence (Prop. we proved it)

$\exists M$ s.t

$\forall n \quad -M \le s_n \le M$

Define $S = \{ x \in \mathbb{R} \mid \exists N \; \forall n \ge N \;\; s_n \ge x \}$ (given x)



$S \ne \emptyset$ since $-M \in S$

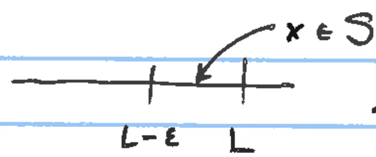
S bdd above by M.

$\exists \sup S = L \in \mathbb{R}$. by LUB property.

<u>We Want to show $L = \lim s_n$.</u>

Let $\varepsilon > 0$ be given.

$L - \varepsilon$ is not an upper bd for S (def of sup.)



$\exists x \in S \qquad L - \varepsilon < x \le L$

(1) $\exists N_1, \forall n \ge N_1, \; s_n \ge x$. (def of S)

$L - \varepsilon < x \le s_n$

$L - \varepsilon < s_n$

<u>We Want</u> to show (2) $\exists N_2 \; \forall n \ge N_2 \qquad s_n < L + \varepsilon$ PTO $\xrightarrow[\text{proof}]{\text{for (2)}}$

(1) & (2) $\Longrightarrow$ $\forall \varepsilon > 0$ $\exists N = \max(N_1, N_2) \quad \forall n \ge N$

once we know both

$L - \varepsilon < s_n < L + \varepsilon$

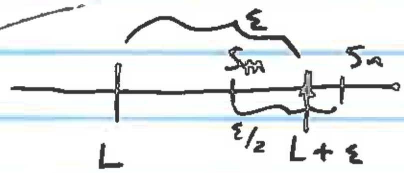
$|s_n - L| < \varepsilon$

$\lim s_n = L$.   So we need to prove (2)

To prove ② $\exists N_2 \ \forall n \geq N_2 \quad s_n < L+\varepsilon$, by contradiction method.

Suppose ② <u>is false</u>

$$\forall N_2 \ \exists n \geq N_2 \qquad s_n \geq L+\varepsilon$$



$\forall$

Defn of Cauchy $\exists N_3 \ \forall n,m \geq N_3 \quad |s_n - s_m| < \frac{\varepsilon}{2}$

$\exists$

Take $N_2 = N_3$. $\quad L+\varepsilon \leq s_n \qquad \exists n \geq N_2$

$$-\frac{\varepsilon}{2} < s_m - s_n < \frac{\varepsilon}{2} \qquad \forall m \geq N_2$$

+

$$L + \frac{\varepsilon}{2} < s_m. \qquad \forall m \geq N_2.$$

$$L + \frac{\varepsilon}{2} \in S$$

Contradiction because $L = \sup S$.

This proves ②

① & ② $\Longrightarrow \lim s_n = L$.