Direct Consequences of Completeness Axiom

ARCHIMEDEAN PROPERTY $\mathbb{N}$ is not bounded in $\mathbb{R}$.

(By Contradiction) Proof Assume: Completeness Axiom.

Suppose $\mathbb{N}$ is bounded.

$\exists\, M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $n \leq M$.

$1 \in \mathbb{N} \neq \emptyset$

By Completeness axiom $\exists \sup(\mathbb{N}) \in \mathbb{R}$.

Let $m = \sup \mathbb{N}$.

$m-1$ is not an upper bd for $\mathbb{N}$,

since m is a least upper bound for $\mathbb{N}$.

$m-1 < m$

$\exists\, n \in \mathbb{N}$ $m-1 < n$ (Defn sup. (ii))

$m < n+1$, $n+1 \in \mathbb{N}$.

which shows that m is <u>not</u> an upper bound for $\mathbb{N}$

$m = \sup \mathbb{N}$ says m is an upper bd for $\mathbb{N}$

Contradiction.

We proved that

Completeness Axiom $\Longrightarrow$ Archimedean Property.

Theorem All of the following are equivalent to

(0) Archimedean Property ($\mathbb{N}$ is unbounded)

(a) $\forall z \in \mathbb{R}\ \exists n \in \mathbb{N},\ n > z$

Archimedeas said → (b) $\forall x > 0\ \forall y \in \mathbb{R}\ \exists n \in \mathbb{N}$ s.t. $nx > y$

(c) $\forall x > 0\ \exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < x$.

Proof

$$\begin{array}{ccc} ⓪ & \Longrightarrow & ⓐ \\ \Uparrow & & \Downarrow \\ ⓒ & \Longleftarrow & ⓑ \end{array}$$

(0) ⟹ (a)

Suppose (a) is false $\sim(\forall z \in \mathbb{R}\ \exists n \in \mathbb{N},\ n > z)$

$\exists z \in \mathbb{R}\ \forall n \in \mathbb{N}\quad n \le z$

i.e. z is an upper bd. for $\mathbb{N}$

$\mathbb{N}$ is bounded

which contradicts (0).

(a) ⟹ (b) Let $x > 0,\ y \in \mathbb{R}$ be given

Let $z = \frac{y}{x}$ then by (a)

(Given (a) $\forall z \in \mathbb{R}\ \exists n \in \mathbb{N},\ n > z$.)

so $\exists n \in \mathbb{N}\quad n > z = \frac{y}{x}$.

$nx > y$. $(x > 0)$.

(b) ⟹ (c) $\forall x > 0\ \forall y \in \mathbb{R}\ \exists n \in \mathbb{N}\quad nx > y$

Given $x > 0$ take $y = 1$, $\exists n \in \mathbb{N},\ nx > 1$

$x > \frac{1}{n} > 0$.

(c) ⟹ (0) Assume $\forall x > 0\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$. ⊛⊛

Suppose (0) is false & $\mathbb{N}$ is bounded

$\exists M$ s.t. $\forall n \in \mathbb{N}\quad 0 < n \le M < M+1$

⊛ $\forall n \in \mathbb{N}\quad \frac{1}{n} > \frac{1}{M+1} = x$ (if we choose x in (c))

⊛ & ⊛⊛ are contradicting.

with proof

Exc 3.3. f g k m,

(f) $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$

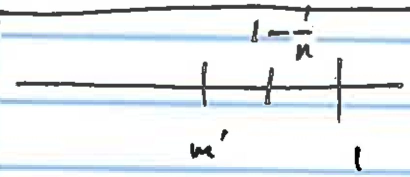
Claim. $\sup S = 1$.

want $\begin{cases} \text{(i)} & \forall s \in S \quad s \leq 1 \\ \text{(ii)} & \forall m' < 1 \quad \exists s \in S \quad s > m' \end{cases}$

(i) $\forall s \in S \quad s = 1 - \frac{1}{n}$ for some n, $n > 0$, $\frac{1}{n} > 0$

$$s = 1 - \frac{1}{n} \leq 1.$$

(ii) Let $m' < 1$ be given....



need $1 - \frac{1}{n} > m'$

need $1 - m' > \frac{1}{n}$

scrapwork

Let $m' < 1$ be given

choose $x = 1 - m' > 0$

$\exists n \in \mathbb{N}$ by (c) of previous thm

s.t. $\frac{1}{n} < 1 - m'$

$m' < 1 - \frac{1}{n} \in S$.

Hence (ii) is established

Exc 3.3 (No proofs for these)

		max	sup	min	inf.
$\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$	g) $\left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$	DNE	1	$\frac{1}{2}$	$\frac{1}{2}$
	k) $\bigcap_{n=1}^{\infty} \overbrace{\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)}^{\text{interval in } \mathbb{R}} = \{1\}$	1	1	1	1
	m) $\{ r \in \mathbb{Q} \mid r < 5 \}$	DNE	5	DNE	DNE
*	$\bigcap_{n=2}^{\infty} \left(-\left(1-\frac{1}{n}\right), 1-\frac{1}{n} \right)$	DNE	$\frac{1}{2}$	DNE	$-\frac{1}{2}$

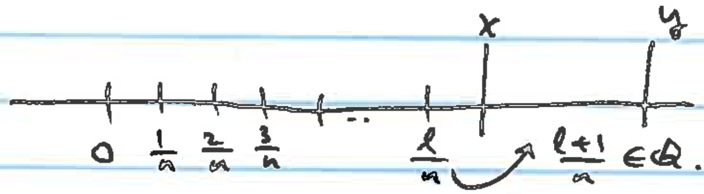
$\left(-\frac{1}{2}, \frac{1}{2}\right)$ $= \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \left(-\frac{2}{3}, \frac{2}{3}\right) \cap \left(-\frac{3}{4}, \frac{3}{4}\right) \cap \cdots \cap \left(-\left(1-\frac{1}{n}\right), 1-\frac{1}{n}\right) \cap \cdots$

⊛⊛⊛ Density of Rationals

<u>Thm:</u> $\forall x, y \in \mathbb{R} \; (x<y \Rightarrow \exists r \in \mathbb{Q},\; x<r<y)$

<u>Proof</u> Case 1 $0<x<y$

<u>IDEA</u>:



if $\frac{1}{n} < \underbrace{y-x}_{\text{distance between } x \,\&\, y.}$

<u>Actual Proof for $0<x<y$</u>

Let $0<x<y$ be given.

$\exists n \in \mathbb{N} \quad n > \frac{1}{y-x}$ (Arch. Principle)

hence. $\frac{1}{n} < y-x$.

Let $S = \{k \in \mathbb{N} \mid k \geq \underset{\text{fixed \#}}{ny}\} \neq \emptyset$ since $\mathbb{N}$ is unbounded (Arch. princ.)

$S \subseteq \mathbb{N}$

$\exists\, m = \min S \in S$ by Well ordering principle. $m \in \mathbb{N}$

<u>Claim</u> $nx \overset{\text{(iii)}}{<} m-1 \overset{\text{(ii)}}{<} ny \overset{\text{(i)}}{\leq} m$

(i) $ny \leq m$ since $m \in S$.

(ii) $m-1 < ny$ (otherwise : $m-1 \geq ny \Rightarrow m$ is not the minimum of S. contradiction.)

(iii) $nx < m-1$: (otherwise :

$$\begin{array}{rl} & m-1 \leq nx \\ + & 1 < n(y-x) \quad (\Leftarrow \tfrac{1}{n} < y-x) \\ \hline & m < ny \end{array}$$

can't happen since $m \in S$.

Hence Claim Holds.

$nx < m-1 < ny \leq m$

$x < \frac{m-1}{n} < y$

$r = \frac{m-1}{n} \in \mathbb{Q}$ since $m, n \in \mathbb{N}$.