Oct 12, 2018

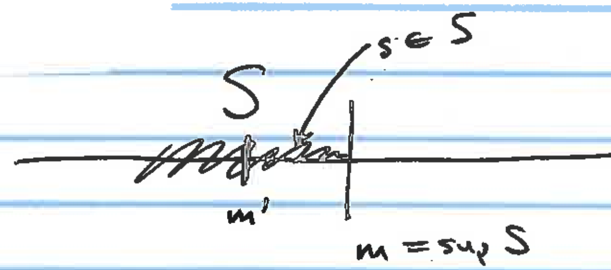
SUPREMUM & LUB PROPERTY / COMPLETENESS AXIOM

Defn Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$, S be bounded above.
A number m is called a supremum or least upper bound for S if

(i) m is an upper bound for S
($\forall s \in S$, $s \le m$)

and

ii) There is no smaller upper bound for S than m
($\Leftrightarrow$ For any $m' < m$, m' is not an upper bd for S)
($\Leftrightarrow$ $\forall m' < m$ $\exists s \in S$, $s > m'$)

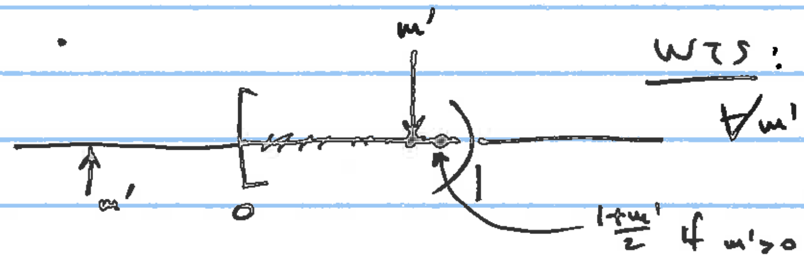


Caution $S = \emptyset$ creates problems.
Then any real number will be an upper bound for $S = \emptyset$.
Least of all real numbers?
$\sup \emptyset = -\infty$ Not we want.

Hw Exc #6 : $\sup(S)$ is a unique # when it exists.

Ex $S=[0,1)$ Show that $\sup(S)=1$.

- $\forall s\in[0,1)$ $0\le s<1$ and hence $s\le 1$.
  1 is an upper bd. for S.

- 

  WTS: $\forall m'<1$, $\exists s\in[0,1)$ s.t. $m'<s$.

Let $m'<1$ be given, then

Case 1 $m'\geq 0$ I can take $s=\frac{1+m'}{2}$

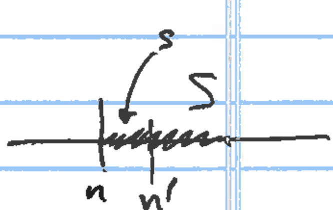
and $0\le m'<1$ implies

$s=\frac{1+m'}{2}>m'$ and $0\le s=\frac{1+m'}{2}<1$. $s\in[0,1)$

Case 2 $m'<0$ take $s=\frac{1}{2}\in[0,1)$

$m'<0<s=\frac{1}{2}$.

Def: Let $S\subseteq\mathbb{R}$, $S\neq\emptyset$, S be bounded below.
A number n is called an infimum or greatest lower bound for S if
(i) n is a lower bound for S
($\Leftrightarrow$ $\forall s\in S$ $n\le s$)
(ii) There is no larger lower bd for S than n.
$\Leftrightarrow$ ($\forall n'>n$, n' is not a lower bd for S)
$\Leftrightarrow$ ($\forall n'>n$ $\exists s\in S$ s.t. $s<n'$)



Def Real numbers is an ordered field containing $\mathbb{Q}$ and satisfying the Completeness Axiom.

COMPLETENESS AXIOM / LEAST UPPER BOUND PROPERTY

For every non-empty subset S of $\mathbb{R}$, that is bounded above has a supremum in $\mathbb{R}$.

$$\forall S \subseteq \mathbb{R} \left(S \neq \emptyset, \text{ and } \exists\, m \in \mathbb{R}\ \forall s \in S \ \ s \leq m \implies \exists \sup(S) \in \mathbb{R}. \right)$$

Ex $S = \left\{ \frac{p}{q} \in \mathbb{Q} \;\middle|\; \left(\frac{p}{q}\right)^2 < 2,\ \frac{p}{q} > 0 \right\}$

(i) $S \subseteq \mathbb{R}$, S is bounded above by $\sqrt{2}$
S is bounded below by 0
$\sup S = \sqrt{2} \in \mathbb{R}$. (requires proof.)
$\inf S = 0 \in \mathbb{R}$. (requires proof.)

(ii) $S \subseteq \mathbb{Q}$. S is bounded above by 1.5 in $\mathbb{Q}$
S is " below by 0 in $\mathbb{Q}$.

There is **NO** $\sup(S)$ in $\mathbb{Q}$.

$\mathbb{Q}$ doesn't satisfy the $\begin{cases} \text{Completeness Axiom/} \\ \text{LUB property.} \end{cases}$

More examples to be done on Monday.