Oct 10, 2018

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Cauchy Sequences (Continue) (4.3)

Ex $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$

For $m<n$; $S_n - S_m = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n}$.

$$S_{2m} - S_m = \underbrace{\frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m}}_{m \text{ of those, each } \geq \frac{1}{2m}} \geq m \cdot \frac{1}{2m} = \frac{1}{2}$$

$\Longrightarrow$ (s_n) is <u>not</u> Cauchy.

Why?

Recall defn Cauchy:

$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n,m \in \mathbb{N} \; (n,m \geq N \Rightarrow |s_n - s_m| < \varepsilon)$

If we take $\varepsilon = \frac{1}{2}$ in the above example we see that

Compare

$\forall m \in \mathbb{N}$ $\quad |S_{2m} - S_m| \geq \frac{1}{2}$

hence $|S_{2m} - S_m| < \frac{1}{2}$ is false $\forall m \in \mathbb{N}$.

We obtained that $\exists \varepsilon = \frac{1}{2} \; \forall N \in \mathbb{N} \; \exists n = 2N, m = N \quad |S_n - S_m| \geq \frac{1}{2} = \varepsilon$

To show: $S_n \to \infty$

$\forall m \in \mathbb{N}$ $\quad S_{2m} - S_m \geq \frac{1}{2}$

$S_{4m} - S_{2m} \geq \frac{1}{2}$

$S_{8m} - S_{4m} \geq \frac{1}{2}$

$S_{2^k m} - S_{2^{k-1} m} \geq \frac{1}{2}$.

Let $m=1$

$$S_{2^k} - S_{2^{k-1}} \geq \frac{1}{2}$$

$$S_{2^{k-1}} - S_{2^{k-2}} \geq \frac{1}{2}$$

$$\vdots$$

$$S_4 - S_2 \geq \frac{1}{2}$$

$$S_2 - S_1 \geq \frac{1}{2}$$

$$S_{2^k} \geq S_1 + k\cdot\frac{1}{2}$$

From Calc I: $[\![x]\!]$ greatest integer function.

if $2^k \leq \ell$.

$$S_\ell \geq S_1 + \frac{1}{2}\overbrace{[\![\log_2 \ell]\!]}^{k} \longrightarrow \infty$$

as $\ell \to \infty$, $k \to \infty$

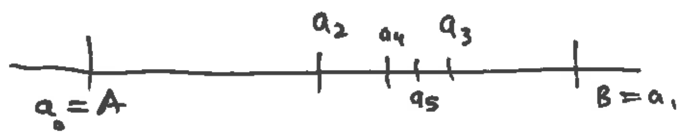
$\Longrightarrow$ $S_\ell \longrightarrow \infty$.

$S_\ell \nearrow$ and unbounded.

Not convergent $\Longleftarrow$ Not Cauchy

Not Cauchy.

we proved convergent $\Rightarrow$ Cauchy



Ex Let $a_0 = A$

$a_1 = B$, $A \neq B$.

Define $a_{n+1} = \dfrac{a_n + a_{n-1}}{2} \quad \forall n \geq 1.$

mid pt of the last two terms

each time we divide to equal halfs. $\longrightarrow$ $|a_{n+1} - a_n| = \dfrac{|a_n - a_{n-1}|}{2}$

$$|a_0 - a_1| = |A - B| = C \quad \text{say}$$

$$|a_1 - a_2| = \frac{C}{2}$$

$$|a_2 - a_3| = \frac{C}{4}$$

$$\vdots$$

$$|a_{n+1} - a_n| = \frac{C}{2^n}$$

Correction

$$|a_n - a_m| \leq \frac{C}{2^n}\underbrace{\left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{m-n-1}}\right)}_{<2} < \frac{2C}{2^n}.$$

$\Longrightarrow$ (a_n) is Cauchy. Why?

$\forall \varepsilon > 0 \ \exists N > \log_2 \frac{\varepsilon}{2C} \quad \forall n, m \geq N$

$$|a_n - a_m| < \underbrace{\frac{2C}{2^N} < \varepsilon}_{N > \log_2 \frac{2C}{\varepsilon}}.$$

CAUTION

False in general (*)(*)(*)

Thm: In $\mathbb{R}$ Cauchy $\Longrightarrow$ convergent. (Caution (a_n) is not monotone.)

Now I know it is convergent

$$L = \lim a_{n+1} = \lim \frac{a_n + a_{n-1}}{2} = \frac{L}{2} + \frac{L}{2} = L$$

$L = L$ no use to determine L.

What is L?

Take $A < B$

$a_0 = A$ $\qquad C = B - A$

$a_1 = A + C$

$a_2 = A + C - \frac{C}{2}$

$a_3 = A + C - \frac{C}{2} + \frac{C}{4}$

$\vdots$

$$L = \lim a_n = A + C\Big(\underbrace{1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots}_{\frac{1}{1-(-\frac{1}{2})} = \frac{1}{3/2} = \frac{2}{3}}\Big)$$

$$= A + C \cdot \frac{2}{3}$$

$$= A + \frac{2}{3}(B - A) = A + \frac{2B}{3} - \frac{2A}{3}$$

Caution: Fixed typos.

$$= \frac{A}{3} + \frac{2B}{3} = \frac{A + 2B}{3}$$

If $B > A$, same arguments give $L = \frac{A + 2B}{3}$ as well.

(3.3) **C** UPPER BOUNDS & SUPREMUM
LOWER BOUND & INFIMUM.

use Capital letters for Sets; lower case for numbers $a \in A$

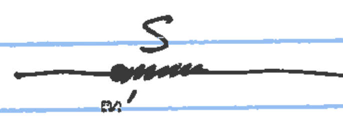
Defn Let $S \subseteq \mathbb{R}$.

- A number m is called an upper bound for S if $\forall s \in S,\ s \le m$
- A number m' is called a lower bound for S if $\forall s \in S\ \ s \ge m'$
- The set S is called bounded if it is bounded above and bounded below, that is: $\exists\, m, m' \in \mathbb{R}\ \forall s \in S\ \ m' \le s \le m$

- A number m is called a maximum of S if (i) $\forall s \in S, s \le m$, and (ii) $m \in S$

notation $m = \max(S)$

- A number m' is called a minimum of S if (i) $\forall s \in S, s \ge m'$, and (ii) $m' \in S$.



Exs (i) $[1,2]$ bounded
upper bounds $100, 3, 2$ many others
lower bounds $0, -1, 1$ " "
$\max [1,2] = 2$
$\min [1,2] = 1$

(i) $\mathbb{N} = \{1, 2, 3, \ldots\}$

Bounded below

$\min \mathbb{N} = 1$.

<u>not</u> Bounded above

not bounded.

(ii) $(-\infty, 0) \subseteq \mathbb{R}$.

Bounded above, upper bounds $0, 5, \pi, \ldots$

$\max(-\infty, 0)$ DNE (Does not exist)

not bounded below.

(iii) $[3, 4)$

Bounded

a lower bd $= 3$

$\min [3, 4) = 3$

an upper bd 5

$\max [3, 4)$ DNE

(v) $S = \{ \frac{p}{q} \mid (p/q)^2 \le 2 \} \subseteq \mathbb{R}$

$\mathbb{Q}$

<u>Best?</u> $\begin{cases} -\sqrt{2} \text{ is a lower bound in } \mathbb{R} \\ \sqrt{2} \text{ is an upper bound in } \mathbb{R}. \end{cases}$

no best choices $\begin{cases} \text{in } \mathbb{Q} \quad 1.5 \text{ an upper bound for } S \\ \text{in } \mathbb{Q} \quad -1.42 \text{ a lower bound for } S \end{cases}$

What do I mean by "best"? Come to class on Fri.