4.3 + 3.3 Continue

Ex

$$S_{n+1} = \frac{S_n^2+1}{2}$$

S_1 given, a fixed starting pt.

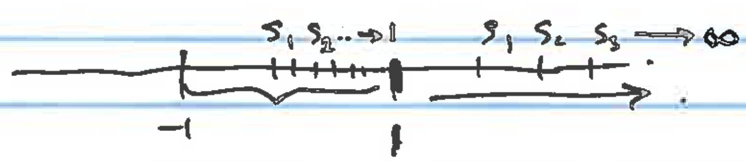
$$S_{n+1} \geq S_n \iff \frac{S_n^2+1}{2} \geq S_n$$

$$\iff S_n^2+1 \geq 2S_n$$

$$\iff S_n^2 - 2S_n + 1 \geq 0$$

$$\iff (S_n - 1)^2 \geq 0 \quad \text{always true.}$$

Regardless of where we start with s_1, $S_n \nearrow$.



Case (a) If $|s_1| \leq 1$ then

$$s_1^2 \leq 1$$

$$1 \leq s_1^2 + 1 \leq 2$$

$$\frac{1}{2} \leq \underbrace{\frac{s_1^2+1}{2}}_{s_2} \leq 1$$

Claim that $|s_n| \leq 1$. Proof by induction. $|s_1| \leq 1$
$p(1)$ is given.

Want $-1 \leq s_k \leq 1 \implies -1 \leq s_{k+1} \leq 1$

$p(k) \Downarrow p(k+1)$.

$$0 \leq s_k^2 \leq 1$$

$$0 \leq \frac{1}{2} = \frac{0+1}{2} \leq \frac{s_k^2+1}{2} \leq \frac{1+1}{2} = 1$$

If $s_1 \in [-1,1]$, then (s_n) a bounded sequence

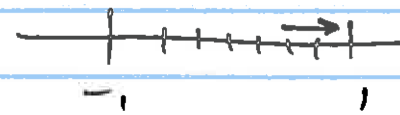
$s_n \nearrow$.

Thm. s_n converges, limit L

$$L = \lim s_{n+1} = \lim \frac{s_n^2+1}{2} = \frac{L^2+1}{2}$$

$$2L = L^2+1$$
$$0 = L^2-2L+1$$
$$1 = L.$$

$$\boxed{\lim s_n = 1}$$



Case (b) If $s_1 > 1$, then we will see that $s_n \to \infty$.

$s_1 > 1 \Rightarrow s_1 = 1+\alpha$. $\alpha > 0$

Claim $s_n \geq 1 + \alpha + (n-1)\frac{\alpha^2}{2}$ $\forall n$. $p(n)$

$p(1)$ $s_1 \geq 1+\alpha$ ✓

To show $p(k) \Rightarrow p(k+1)$:

$$s_k \geq 1 + \alpha + (k-1)\frac{\alpha^2}{2}$$

$$s_{k+1} = \frac{1+s_k^2}{2} \geq \frac{1}{2}\left(1 + 1 + 2\alpha + \alpha^2 + 2\cdot(k-1)\frac{\alpha^2}{2} + (\alpha^3 \cdots)\right)$$

$$= \frac{1}{2}\left(2 + 2\alpha + k\alpha^2 + \cdots\right)$$

$$= 1 + \alpha + \frac{k}{2}\alpha^2 + \underbrace{\cdots}_{+\text{ terms}}$$

$$\geq 1 + \alpha + \frac{k}{2}\alpha^2 \qquad p(k+1). ✓$$

$\forall n \quad s_n \geq 1 + \alpha + (n-1)\frac{\alpha^2}{2}$

s_1 fixed, α is fixed. $\alpha > 0$, $\frac{\alpha^2}{2} > 0$

HW

Thm. 4.2.12

$$\left.\begin{array}{l} \text{As } n \to \infty \quad 1 + \alpha + (n-1)\frac{\alpha^2}{2} \to \infty. \\ \implies s_n \to \infty \end{array}\right.$$

$\lim s_n = +\infty.$

Case (c) $\quad \left.\begin{array}{l} s_1 \in (-\infty, -1) \\ s_{n+1} = \dfrac{s_n^2+1}{2} \end{array}\right\} \Rightarrow s_2 > 1$

Hence $s_n \to +\infty$.

Thm

Let (s_n) be monotone, increasing, unbounded,

Then $\lim_{n\to\infty} s_n = +\infty$.

Proof $\quad s_1 \leq s_2 \leq s_3 \leq \cdots$

bounded below.

unbounded $\implies$ not bounded above.

$$\text{not}\underbrace{\left(\exists M \; \forall n \in \mathbb{N} \quad s_n \leq M\right)}$$

$\forall M \; \exists n_0 \in \mathbb{N} \quad s_{n_0} > M.$

$\forall k \in \mathbb{N} \quad s_{n_0+k} \geq \cdots \geq s_{n_0+1} \geq s_{n_0} > M$

$\forall M \; \exists n_0 \; \forall k \geq 0 \quad k + n_0 \geq n_0 \Rightarrow s_{n_0+k} > M$

$\lim s_n = +\infty.$ by Defn 4.2.9

B Cauchy sequences

<u>Defn</u> A sequence (s_n) is called Cauchy,

if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n, m \in \mathbb{N}$

$$(n, m \geq N \Rightarrow |s_n - s_m| < \varepsilon)$$

<u>Recall</u> <u>Defn</u> A sequence (s_n) is called convergent if

$$\exists L \in \mathbb{R}, \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \in \mathbb{N} \ (n \geq N \Rightarrow |s_n - L| < \varepsilon$$

<u>Prop</u> Every Convergent sequence is Cauchy

<u>Proof</u>

Let $\varepsilon > 0$ be given. Since $\lim s_n = L$

$\exists N \ \forall n \in \mathbb{N} \ (n \geq N \Rightarrow |s_n - L| < \frac{\varepsilon}{2})$

Then $\forall n, m \geq N$

$$|s_n - s_m| = |s_n - L + L - s_m| \leq |s_n - L| + |L - s_m|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \#$$

* <u>Converse</u> : <u>Big question</u>

In $\mathbb{Q}$, converse is false Cauchy $\not\Rightarrow$ convergent

*** <u>Thm</u>: In $\mathbb{R}$, every Cauchy sequence converges.
(<u>that is</u> completeness.)

Ex s_n = truncation of $\sqrt{2}$ at n^{th} decimal place
the decimal expansion of

$$s_0 = 1. \quad \in \mathbb{Q}$$
$$s_1 = 1.4 = \frac{14}{10} \quad \in \mathbb{Q}$$
$$s_2 = 1.41 = 141/100 \in \mathbb{Q}.$$
$$s_3 = 1.414 = 1414/1000 \in \mathbb{Q}.$$
$$\vdots$$

$s_n \to \sqrt{2}$ in $\mathbb{R}$.

s_n doesn't converge to anything in $\mathbb{Q}$.

Is (s_n) Cauchy? YES. in $\mathbb{Q}$
YES in $\mathbb{R}$

$|s_n - s_m|$ estimatation

$m > n$.

Ex $s_2 = 1.41$
$s_5 = 1.41421$
$s_5 - s_2 = 0.00421 \leq 0.01$
$|s_5 - s_2| \leq 10^{-2}$

$$|s_n - s_m| \leq 10^{-n} < \varepsilon \quad \text{if we choose}$$
$m > n$

$$N > -\log_{10}\varepsilon > 0$$
if $0 < \varepsilon < 1$

$\forall n, m \geq N \quad |s_n - s_m| < \varepsilon.$

Ex $n \in \mathbb{N}$, $\frac{\sqrt{2}}{n} \notin \mathbb{Q}$. $\quad \frac{\sqrt{2}}{n} \to 0 \in \mathbb{Q}$.

s_n above $\quad s_n \in \mathbb{Q}$. $\quad \lim s_n = \sqrt{2} \notin \mathbb{Q}$.

Lemma: Every Cauchy sequence is bounded.

Proof: Let $\varepsilon = 1$ be given

$\exists N \in \mathbb{N} \quad \forall n, m \in \mathbb{N}$

$$(n, m \geq N \implies |s_n - s_m| < 1).$$

particularly $\forall m \geq N \quad |s_N - s_m| < 1.$

$$\forall m \geq N \qquad |s_m| \leq |s_N| + 1$$

Let $M = \max(|s_1|, |s_2|, |s_3|, \ldots \; |s_{N-1}|, |s_N| + 1)$

$\forall m \in \mathbb{N} \qquad |s_m| \leq M$ since

when $m < N$ then $|s_m| \leq |s_m| \leq M$

when $m \geq N$ $\qquad |s_m| \leq |s_N| + 1$

$$\begin{tikzcd}
\text{Convergent} \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "\text{done earlier.}"'] & \text{Cauchy} \arrow[d, Rightarrow] \\
& \text{Bounded}
\end{tikzcd}$$