Oct 5, 2018

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3.3 + 4.3

Completeness of $\mathbb{R}$.

Thm: In an ordered field containing $\mathbb{Q}$, the following are equivalent:

Every bounded monotone sequence converges $\Longleftrightarrow$ Every Cauchy sequence converges

Every bounded monotone sequence converges $\Longleftrightarrow$ Least Upper Bound Property holds

Every Cauchy sequence converges $\Longleftrightarrow$ Least Upper Bound Property holds

1. We assume one of them as the axiom of completeness of $\mathbb{R}$.
2. We prove the other two as theorems.
3. Hence in $\mathbb{R}$, all 3 are true
4. in $\mathbb{Q}$ all 3 are false

In this textbook we will take LUB property as the axiom of completeness.

In General context "Cauchy sequences Converge" is usually taken for completeness.

A Monotone sequences

Defn A sequence (s_n) is called increasing if $s_{n+1} \geq s_n \ \forall n \in \mathbb{N}$.

A sequence (s_n) is called decreasing if $s_{n+1} \leq s_n \ \forall n \in \mathbb{N}$.

A sequence is called monotone if it is either an increasing sequence, or a decreasing seq.

Ex. $\left(\frac{n}{n+1}\right)$ $\quad \frac{1}{2} \leq \frac{2}{3} \leq \frac{3}{4} \leq \frac{4}{5} \leq \cdots$ increasing

$\left(e^{-1/n}\right)$

$n \uparrow$

$\frac{1}{n} \downarrow$

$-\frac{1}{n} \uparrow$

$e^{-\frac{1}{n}} \uparrow$.

Thm: Every Bounded monotone sequence converges

- This theorem we will prove after learning about LUB property
- This theorem has many applications.

Ex 1. $\begin{cases} S_1 = 2 \\ S_{n+1} = \frac{1}{4}(s_n + 5) \ \forall n \geq 1 \end{cases}$

2 · Prove monotone

1 · Prove bounded

3 · Find limit.

$2, \frac{7}{4}, \frac{27}{16}, \ldots$

1 Claim: $0 \leq S_n \leq 3$ $\quad p(n)$

Proof by induction.

$p(1)$ $\quad 0 \leq S_1 = 2 \leq 3$ ✓

Next $p(k) \Longrightarrow p(k+1)$:

$0 \leq S_k \leq 3$ $\quad p(k)$

$5 \leq S_k + 5 \leq 8$ $\quad$ Induction step.

$\frac{5}{4} \leq \underbrace{\frac{S_k+5}{4}} \leq 2$

$0 \leq \frac{5}{4} \leq S_{k+1} \leq 2 \leq 3.$ $\quad p(k+1)$ ✓

By Thm of Induction

$\forall n \in \mathbb{N} \quad 0 \leq S_n \leq 3.$

2 To prove monotone:

Claim: $q(n)$: $\quad S_{n+1} \leq S_n$

$q(1)$ $\quad \frac{7}{4} = S_2 \leq S_1 = 2$ ✓

Induction step $\quad$ Assume $q(k)$ $\quad S_{k+1} \leq S_k$

$S_{k+1} + 5 \leq S_k + 5$

$\forall k$ $\quad S_{k+2} \leq \frac{1}{4}(S_{k+1} + 5) \leq \frac{1}{4}(S_k + 5) = S_{k+1}$

$q(k+1)$ $\quad S_{k+2} \leq S_{k+1}$ $\quad (q(k+1))$

Thm of Induction $\Rightarrow \forall n \quad S_{n+1} \leq S_n.$

s_n is a bounded & monotone sequence.

Thm (*) $\Longrightarrow$ s_n is a convergent sequence.

Let $\lim s_n = L. \in \mathbb{R}$.

$$L = \lim s_{n+1} = \lim \frac{1}{4}(s_n+5) = \frac{1}{4}(L+5)$$

$$L = \frac{1}{4}(L+5)$$

$$4L = L+5$$

$$3L = 5$$

$$L = 5/3.$$

Ex $s_n = 1 - s_{n-1} \quad \forall n \in \mathbb{N}$

$s_1 = 2.$

Suppose you didn't check convergence

lim of both sides

$$L = \lim s_n = \lim (1 - s_{n-1}) = 1 - L$$

$$L = 1 - L$$

$$2L = 1$$

$$L = \frac{1}{2} \quad \text{No!}$$

Look at

s_n : $2, -1, 2, -1, 2, -1, 2, -1, \ldots$ Divergent

Lesson: Check the convergence first,
then do $\lim s_n$ after that.
Otherwise you will give strange answers.

Ex 2 $\begin{cases} S_{n+1} = \sqrt{2S_n + 1} & \text{for } n \geq 1. \\ S_1 = 2. \end{cases}$ $2, \sqrt{5}, \sqrt{2\sqrt{5}+1},$

$\sqrt{\sqrt{2\sqrt{5}+1}\cdot 2 + 1}, \ldots$

- Show $S_{n+1} \geq S_n$.
- Show bdd
- Show limit is $1+\sqrt{2}$.

• Show bdd. Claim $0 \leq S_n \leq 3$ $p(n)$.

$p(1)$ $0 \leq S_1 = 2 \leq 3$ ✓

To show $p(k) \Rightarrow p(k+1)$:

$0 \leq S_k \leq 3$ $p(k)$

$0 \leq 2S_k \leq 6$

$1 \leq 2S_k + 1 \leq 7$

$0 \leq 1 \leq \sqrt{2S_k+1} \leq \sqrt{7} \leq 3.$

$0 \leq S_{k+1} \leq 3.$ $p(k+1)$

Thm of Induction $\Rightarrow \forall n \in \mathbb{N}$ $0 \leq S_n \leq 3$

• Try increasing $0 \leq S_n \leq S_{n+1}$ $q(n)$

$q(1)$ $0 \leq 2 = S_1 \leq \sqrt{5} = S_2$

$q(k)$ $0 \leq S_k \leq S_{k+1}$

$0 \leq 2S_k \leq 2S_{k+1}$

$1 \leq 2S_k + 1 \leq 2S_{k+1} + 1$

$S_{k+1} = \sqrt{2S_k+1} \leq \sqrt{2S_{k+1}+1} = S_{k+2}$

$q(k+1)$ $0 \leq S_{k+1} \leq S_{k+2}$

Thm Induction $\Rightarrow$ $\forall n$ $0 \leq S_n \leq S_{n+1}$.

(s_n) is a bounded monotone sequence.

Thm (*) $\lim s_n$ exists

Let $\lim s_n = L$

$$L = \lim s_{n+1} = \lim \sqrt{2s_n+1} = \sqrt{2L+1}$$

$$L = \sqrt{2L+1} \geqslant 0$$

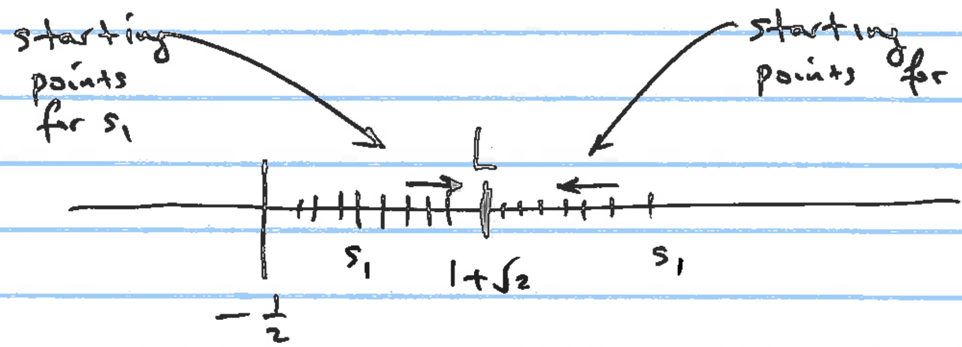
$$L^2 = 2L+1$$

$$L^2 - 2L - 1 = 0$$

$$L = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

$$L = 1+\sqrt{2} \geqslant 0.$$

(b) $\left.\begin{aligned} s_1 &= 3 \\ s_{n+1} &= \sqrt{2s_n+1} \end{aligned}\right\} \Rightarrow \quad s_n \downarrow . \; 1+\sqrt{2}.$



Behaviour of s_n depending on where s_1 is chosen.

$$\left\{\begin{array}{l} \text{If } s_1 \in [-\tfrac{1}{2}, 1+\sqrt{2}), \text{ then } s_n \nearrow 1+\sqrt{2} \\ \text{if } s_1 \in (1+\sqrt{2}, \infty), \text{ then } s_n \searrow 1+\sqrt{2} \\ \text{if } s_1 = 1+\sqrt{2}, \text{ then } s_n = 1+\sqrt{2} \ \forall n \\ \text{if } s_1 = -\tfrac{1}{2}, \text{ then } s_2 = 0, s_3 = 1, s_4 = \sqrt{3} \ldots \longrightarrow 1+\sqrt{2} \\ \text{if } s_1 < -\tfrac{1}{2} \quad s_2 \text{ DNE} \quad \text{no sequence} \end{array}\right.$$