Oct 3, 2018

1

4.2 Continue

Ex $\lim_{n\to\infty} \frac{n^2+7n}{3n^2+6} = \lim_{n\to\infty} \frac{n^2(1+\frac{7}{n})}{n^2(3+\frac{6}{n^2})}$

$= \lim_{n\to\infty} \frac{1+\frac{7}{n}}{3+\frac{6}{n^2}} \underset{\text{Thm 4.2.1}}{=} \frac{\lim 1+\frac{7}{n}}{\lim 3+\frac{6}{n^2}} = \frac{1}{3}$

Convergent

$\lim_{n\to\infty} 1+\frac{7}{n} = 1+7\cdot 0$ since we know $\lim_{n\to\infty}\frac{1}{n}=0$

$\lim_{n\to\infty} 3+\frac{6}{n^2} = 3+7\cdot 0$ $\qquad \lim_{n\to\infty}\frac{1}{n^2}=0$

$\forall \varepsilon>0\ \exists N>\frac{1}{\sqrt{\varepsilon}}$ $\left|\frac{1}{n^2}-0\right| \le \frac{1}{N^2} < \varepsilon$ $\forall n\ge N$

Thm 4.2.4 Let s_n, t_n be convergent sequences s.t.

$$\left.\begin{array}{l} \lim s_n = s \\ \lim t_n = t \\ s_n \ge t_n \ \forall n \end{array}\right\} \Longrightarrow s \ge t$$

Ex.

$$\left.\begin{array}{l} s_n = \frac{1}{n} \\ t_n = 0 \\ \frac{1}{n} > 0 \end{array}\right\} \not\Longrightarrow \lim s_n = 0 > 0$$

assuming strict inequality

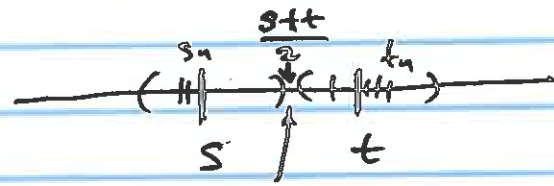
false

But Don't expect strict inequali

Thm 4.2.4 <u>Proof</u>: Proof by Contradiction.

$$\left.\begin{aligned} &\lim s_n = s \\ &\lim t_n = t \\ \circledast\ &s_n \geq t_n \ \forall n. \end{aligned}\right\} \text{Hypothesis}$$

$$s < t \quad \} \text{ negation of conclusion}$$



Take $\varepsilon = \frac{t-s}{2}$

Let $\varepsilon = \frac{t-s}{2} > 0$

$$\left.\begin{aligned} &\exists N_1 \quad \forall n \in \mathbb{N} \ (n \geq N_1 \Rightarrow |s_n - s| < \varepsilon) \\ &\exists N_2 \quad \forall n \in \mathbb{N} \ (n \geq N_2 \Rightarrow |t_n - t| < \varepsilon) \end{aligned}\right\}$$

Let $N = \max(N_1, N_2)$, $\forall n \geq N$:

$|s_n - s| < \varepsilon \Rightarrow -\varepsilon < s_n - s < \varepsilon$

$\Rightarrow s - \varepsilon < s_n < s + \varepsilon = \frac{s+t}{2}$

$|t_n - t| < \varepsilon \Rightarrow -\varepsilon < t_n - t < \varepsilon$

$\frac{s+t}{2} = t - \varepsilon < t_n < t + \varepsilon$

$\forall n \geq N \quad s_n < \frac{s+t}{2} < t_n$; but $s_n \geq t_n \ \forall n.$ $\circledast$ above

Contradiction

Hence $s < t$ is false.

Conclusion $s \geq t$.

4.2.6

Ex 6 p. 178 True/False; prove/counterex.

(c) is true by Thm 4.2.1

$$\left.\begin{array}{l} s_n \to s \\ s_n + t_n \to L. \end{array}\right\} \Longrightarrow -s_n \to -s \quad \text{Thm 4.2.1 (b)}$$

$$t_n = s_n + t_n + (-s_n) \longrightarrow L - s$$

Thm 4.2.1 (a)

counterex.

(a) $s_n = -n$ diverges

$t_n = n$ diverges

$s_n + t_n = 0$ converges

(b) $s_n = (-1)^n$ divergent

$t_n = (-1)^n$ divergent

$s_n \cdot t_n = 1$ convergent

(d) $s_n = 0 \longrightarrow 0$

$t_n = n$. divergent.

$s_n t_n = 0 \longrightarrow 0$

Infinite Limits

Defn A sequence (s_n) is said to diverge to ∞ if

$$\forall M \ \exists N \ \forall n \in \mathbb{N} \ (n \geq N \Rightarrow s_n > M)$$

Notation: $s_n \to \infty$, $\lim_{n\to\infty} s_n = \infty$

Ex

$s_n = n^3$, show that $\lim_{n\to\infty} n^3 = +\infty$

$\forall M$: $\exists N \in \mathbb{N}$ $N > \sqrt[3]{M+1}$

$\forall n \geq N$ $s_n = n^3 \geq N^3 > M+1 > M$

Ex $\lim_{n\to\infty} \frac{6n^2+7}{n-1} = \infty$, prove it

Scrap work

Notproof

$$\frac{6n^2+7}{n-1} \geq 6n \ ?$$

$$6n^2+7 \geq 6n(n-1) \ ?$$

$$\cancel{6n^2}+7 \geq \cancel{6n^2} - 6n$$

$$7 \geq \underbrace{-6n}_{0<}$$

may be able to write a proof

PTO for proof.

<u>Proof of</u> $\lim_{n\to\infty} \frac{6n^2+7}{n-1} = \infty$.

$\forall n \in \mathbb{N} \quad 7 \geq 0 > -6n$.

$$6n^2+7 \geq 6n^2 - 6n = 6n(n-1)$$

if $n \geq 2$ $$\frac{6n^2+7}{n-1} \geq 6n.$$

Let $M \in \mathbb{R}$ be given choose $N \in \mathbb{N}$ s.t $N > \frac{M}{6}$ and $N \geq 2$

Hint ←

$\forall n \geq N \quad \frac{6n^2+7}{n-1} \geq 6n \geq 6N > M$

How to prove.

<u>Thm</u> $s_n \to +\infty$, $\forall n \; t_n \geq s_n \Rightarrow t_n \to \infty$

$t_n \to -\infty$, $\forall n \; t_n \geq s_n \Rightarrow s_n \to -\infty$.

Thm $\forall n \; s_n > 0$.

$$\left(s_n \to \infty \iff \frac{1}{s_n} \to 0 \right)$$

Shows the necessity of $s_n > 0$ for $\Leftarrow$.

Ex $\frac{1}{s_n} = t_n = \frac{(-1)^n}{n} \to 0$

$$\underbrace{s_n = \frac{1}{t_n} = n \cdot (-1)^n}_{\text{divergent}} \;\not\to \infty, \;\not\to -\infty$$

$s_n = \frac{1}{t_n}$

Proof: ($\Rightarrow$:) Assume $s_n > 0 \; \forall n$

$s_n \to \infty$

We want $\frac{1}{s_n} \to 0$.

$s_n \to \infty$

given (*) $\forall M \in \mathbb{R} \; \exists N \; \forall n \geq N \quad s_n > M$

$\frac{1}{s_n} < \frac{1}{M}$

Let $\varepsilon > 0$ be given take $M = \frac{1}{\varepsilon}$

then choose N according to (*) above

$\forall \varepsilon > 0 \; \exists N \; \forall n \in \mathbb{N} \; n \geq N$

$$\left| \frac{1}{s_n} - 0 \right| = \left| \frac{1}{s_n} \right| = \frac{1}{s_n} < \frac{1}{M} = \varepsilon$$

($\Leftarrow$). next page. PTO

Proof:

($\Leftarrow$:) Assume $\frac{1}{s_n} \to 0$, $s_n > 0$

Want: $s_n \to \infty$.

Let $M \in \mathbb{R}$ be given,

Choose $\varepsilon = \frac{1}{\max(M,1)} > 0$ (in case $M \leq 0$)

$\frac{1}{s_n} \to 0 \implies \exists N \in \mathbb{N} \ \forall n \geq N$

$$\left| \frac{1}{s_n} - 0 \right| = \frac{1}{s_n} < \varepsilon \quad (s_n > 0)$$

$$s_n > \frac{1}{\varepsilon} = \max(M,1) \geq M$$

We showed that

$$\forall M \in \mathbb{R} \ \exists N \ \forall n \in \mathbb{N} \ (n \geq N \implies s_n > M).$$

$$\lim_{n\to\infty} s_n = +\infty.$$