4.2 Continue

p171 Thm 4.2.1 (c)

Let $s_n \to s$

$t_n \to t$.

Then $s_n t_n \to st$.

Proof Recall $s_n \to s \Rightarrow s_n$ is bounded.

$\Rightarrow \exists N_0 \; \forall n \in \mathbb{N} \; |s_n| \le N_0$

Let $M = \max(N_0, |t|, 1) > 0$

$\forall \varepsilon > 0 \; \exists N_1 \in \mathbb{N} \; \forall n \in \mathbb{N} \left(n \ge N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2M} \right)$

$\forall \varepsilon > 0 \; \exists N_2 \in \mathbb{N} \; \forall n \in \mathbb{N} \left(n \ge N_2 \right) \Rightarrow |t_n - t| < \frac{\varepsilon}{2M})$

Given $\varepsilon > 0$, choose N_1, N_2 as above, and let

$N = \max(N_1, N_2)$

$$
\begin{aligned}
\forall n \ge N, \; |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&= |s_n(t_n - t) + t(s_n - s)| \\
&\le |s_n||t_n - t| + |t||s_n - s| \\
&\le M|t_n - t| + M|s_n - s| \\
&< M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon .
\end{aligned}
$$

We established $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N} \left(n \ge N \Rightarrow |s_n t_n - st| < \varepsilon \right)$

$\Rightarrow$. $\lim s_n t_n = st$.

#

Thm 4.2.1 (d)

Let $s_n \to s$, $t_n \to t$, where $t_n \neq 0$ $\forall n$ and $t \neq 0$

Then $\frac{s_n}{t_n} \to \frac{s}{t}$.

Scrap work:

$\left| \frac{s_n}{t_n} - \frac{s}{t} \right.$

STEP 2 Then we can use product $s_n\left(\frac{1}{t_n}\right) \to s \cdot \frac{1}{t}$

First: What about an easier case:

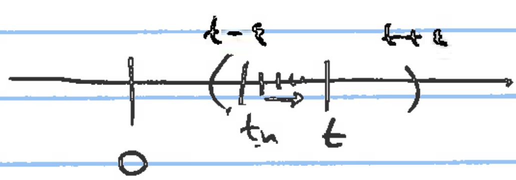
STEP 1 $s_n = 1$, $t_n \to t$ to show $\frac{1}{t_n} \to \frac{1}{t}$

want this small $\left| \frac{1}{t_n} - \frac{1}{t} \right| = \left| \frac{t - t_n}{t t_n} \right| = \frac{|t - t_n|}{|t|\,|t_n|} = |t - t_n| \cdot \frac{1}{|t|} \cdot \frac{1}{|t_n|}$

fixed varying.

we know $|t_n| \le$ bdd.

what about $\left| \frac{1}{t_n} \right| \le$ bdd ?



Trick: keep t_n close t, and away from 0.

$t_n \to t$ For $\varepsilon = \frac{|t|}{2}$, $\exists N_1$ $\forall n \ge N_1$

reverse Δ-inequality

$\underbrace{\big| |t_n| - |t| \big|} \le |t_n - t| < \frac{|t|}{2} = \varepsilon$

One of them is useful. ① $|t_n| - |t| < \frac{|t|}{2}$ and ② $|t| - |t_n| < \frac{|t|}{2}$. Which one?

Scrapwork: continues

$$\frac{|t|}{2} = |t| - \frac{|t|}{2} \overset{②}{<} |t_n| \overset{①}{<} \frac{|t|}{2} + |t| = \frac{3}{2}|t|$$

$$\underbrace{\frac{2}{|t|} > \left|\frac{1}{t_n}\right|}_{\text{need this one:}} > \frac{2}{3}|t|.$$

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| \le \frac{|t - t_n|}{|t||t_n|} \le |t - t_n| \cdot \frac{1}{|t|}\frac{2}{|t|} < \varepsilon$$

want

so we need $|t - t_n| < \frac{\varepsilon |t|^2}{2}$

PTO for actual clean proof.

Thm 4.2.1 (d)

<u>Proof</u> <u>Step 1</u> we will prove $\frac{1}{t_n} \to \frac{1}{t}$, first.
(provided that $t_n \neq 0$ $\forall n$, $t \neq 0$)

For $\varepsilon = \frac{|t|}{2} > 0$ $\exists N_1$ $\forall n \geq N_1$

$$\big|\,|t_n| - |t|\,\big| \leq |t_n - t| < \varepsilon = \frac{|t|}{2}.$$

$$|t_n| - |t| < \frac{|t|}{2} \quad \text{and} \quad |t| - |t_n| < \frac{|t|}{2}$$

$$0 < \frac{|t|}{2} = |t| - \frac{|t|}{2} < |t_n| < |t| + \frac{|t|}{2} = \frac{3}{2}|t|$$

$$\circledast \quad \frac{2}{|t|} > \left|\frac{1}{t_n}\right| \quad \forall n \in \mathbb{N},\ n \geq N_1$$

Let $\varepsilon > 0$ be given choose N_2,

$$\forall n \geq N_2 \quad \underset{**}{|t - t_n| < \frac{\varepsilon |t|^2}{2}} \quad (t \neq 0)$$

$\forall n \geq \max(N_1, N_2) = N$

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \left|\frac{t_n - t}{t\,t_n}\right| = \frac{|t_n - t|}{|t||t_n|} \underset{\circledast}{<} \frac{|t_n - t|}{|t| \cdot \frac{|t|}{2}}$$

$$= \frac{|t_n - t| \cdot 2}{|t|^2} \underset{**}{<} \frac{\varepsilon |t|^2}{2} \cdot \frac{2}{|t|^2} = \varepsilon$$

$\forall \varepsilon > 0\ \exists N \in \mathbb{N}$, $\forall n \geq N$ $\left|\frac{1}{t_n} - \frac{1}{t}\right| < \varepsilon$. Hence $\frac{1}{t_n} \to \frac{1}{t}$.

STEP 2
General Case: $\frac{s_n}{t_n} = s_n \cdot \frac{1}{t_n} \longrightarrow s \cdot \frac{1}{t}$ by STEP 1 & Thm 4.2.1(c

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Exc 15a

Prove that $\lim_{n \to \infty} \left(\sqrt{n+1} - \sqrt{n} \right) = 0$

preparation

$$\left| \sqrt{n+1} - \sqrt{n} \right| = \left| \left(\sqrt{n+1} - \sqrt{n} \right) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right|$$

$$= \left| \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \underset{n \geq N}{<} \frac{1}{\sqrt{N}} \underset{\text{need}}{<} \varepsilon$$

Proof Let $\varepsilon > 0$ be given choose $N \in \mathbb{N}$ s.t.

$N > \frac{1}{\varepsilon^2}$ then $\forall n \geq N,\ n \in \mathbb{N}$

$$\Big| \sqrt{n+1} - \sqrt{n} - \underbrace{0}_{\text{Limit}} \Big| = \left| \left(\sqrt{n+1} - \sqrt{n} \right) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right|$$

$$= \left| \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon .$$

since $N > \frac{1}{\varepsilon^2}$.