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p120 Exc 4.1.13 Squeeze Thm

Let $(a_n), (b_n), (c_n)$ be sequences s.t.

$$a_n \le b_n \le c_n \qquad \forall n \ge N_0 \text{ for some } N_0$$

Assume $\lim a_n = L = \lim c_n$.

Then

$$\lim_{n\to\infty} b_n \text{ exists } \& = L.$$

Proof

(Hypo) $\lim a_n = L$

$\forall \varepsilon > 0\ \exists N_1\ \forall n \in \mathbb{N}\ (n \ge N_1 \Rightarrow |a_n - L| < \varepsilon)$

$L - \varepsilon < a_n < L + \varepsilon$

(Hypo) $\lim c_n = L$

$\forall \varepsilon > 0\ \exists N_2\ \forall n \in \mathbb{N}\ (n \ge N_2 \Rightarrow |c_n - L| < \varepsilon$

$L - \varepsilon < c_n < L + \varepsilon$

For any given $\varepsilon > 0$, choose $N = \max(N_1, N_2, N_0)$

$\forall n \in \mathbb{N},\ n \ge N$

$$L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon$$

$$L - \varepsilon < b_n < L + \varepsilon$$

$$|b_n - L| < \varepsilon$$

$\forall \varepsilon > 0\ \exists N\ \forall n \in \mathbb{N}\ (n \ge N \Rightarrow |b_n - L| < \varepsilon)$

$\lim b_n = L$.

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Prop 4.1.8

Let (s_n), (a_n) be sequences of real #s, $s \in \mathbb{R}$.

If $\exists\, k \in \mathbb{R}$ (fixed real #)

$\exists m \;\; \forall n \geq m \quad |s_n - s| \leq k|a_n|$,

and $\lim a_n = 0$.

$\implies \lim s_n = s$.

Different Proof (from textbook) $\quad -k|a_n| \leq s_n - s \leq k|a_n|$

Squeeze Thm $\implies$ $\begin{cases} \lim a_n = 0 \implies \lim k|a_n| \to 0 \\ \lim s_n - s = 0 \implies \lim s_n = s. \end{cases}$

Actually used 2 Thms from 4.2 (4.2.1 a b)

Thm $\quad$ Exc 7.f p170

If $|x| < 1$, then $\lim\limits_{n\to\infty} x^n = 0$

Proof.

Case 1 $\quad$ If $x = 0$, then $\lim\limits_{n\to\infty} 0^n = 0$ (obvious)

Case 2 $\quad$ If $0 < x < 1$ (x is a fixed #)

Case 3 $\quad$ If $-1 < x < 0$ (" " " " ")

Case 2

$0 < x < 1$ (x is a fixed #)

$1 < \frac{1}{x} = 1 + \alpha$ where $\alpha > 0$

$\forall n \in \mathbb{N}$ $\left(\frac{1}{x}\right)^n = (1+\alpha)^n = 1 + n\alpha + \frac{n(n-1)}{2}\alpha^2 + \cdots$

$\forall n \in \mathbb{N}$ $\frac{1}{x^n} = \left(\frac{1}{x}\right)^n \geqslant n\alpha > 0$

$0 \leq x^n \leq \frac{1}{n\alpha}$. x fixed, α fixed, $n \to \infty$.

$\downarrow \qquad \downarrow$

$0 \qquad 0$

By Squeeze Thm $\lim x^n = 0$.

Case 3 $-1 < x < 0$

$0 < |x| < 1$ $|x|^n = |x^n|$

by Case 2 $\lim\limits_{n\to\infty} |x|^n = 0 = \lim\limits_{n\to\infty} |x^n|$

Exc 9c which we proved!

$\lim x^n = 0$

4.2 Limit Theorems:

Thm 4.2.1

(a) $\left.\begin{array}{l}\lim s_n = s \\ \lim t_n = t\end{array}\right\} \implies \lim\limits_{n\to\infty} (s_n + t_n) = s + t$

Proof

$\lim s_n = s$ (given)

① . $\forall \varepsilon > 0\ \exists N_1\ n \in \mathbb{N}\ \left(n \geq N_1 \implies |s_n - s| < \frac{\varepsilon}{2}\right)$.

$\lim t_n = t$ (given)

② . $\forall \varepsilon > 0\ \exists N_2\ n \in \mathbb{N}\ \left(n \geq N_2 \implies |t_n - t| < \frac{\varepsilon}{2}\right)$.

Let $\varepsilon > 0$ be given.

Choose N_1 as in ①

Choose N_2 as in ②

choose $N = \max(N_1, N_2)$

$\forall n \in \mathbb{N}$, $n \geq N$, one has

$$|(s_n + t_n) - (s+t)| = |(s_n - s) + (t_n - t)|$$
$$\leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence:

$\forall \varepsilon > 0\ \exists N\ \forall n \in \mathbb{N}\ \big(n \geq N \implies |(s_n + t_n) - (s+t)| < \varepsilon\big)$