Ex

$\forall x \exists y \forall z \quad (x^2+y^2)z = 0$ False

$\underbrace{\exists x \exists y \forall z \quad (x^2+y^2)z = 0}$ True

$\exists\ x = 0 = y \implies (x^2+y^2)z = 0$

$\forall x$, say $x = 1$ is given $\quad (\underbrace{1^2+y^2}_{>0})z \neq 0$ one can take $z \neq 0$ since $\forall z$

4.1 Continue:

Ex Prove that $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

Scrapwork: $\left| \frac{1}{2^n} - 0 \right| = \frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

$s_n \quad L \quad \forall n \geq N \quad$ choose $N > \frac{1}{\varepsilon}$

<u>if</u> $2^n \geq n$

a proof?

use induction $n=1 \quad 2^1 \geq 1$.

?? $2^n \geq n \implies 2^{n+1} \geq n+1$

<u>Want</u> $2^{n+1} = 2^n \cdot 2 \geq n \cdot 2 \geq n+1$

add n

$n \geq 1$

no proof, but it's a plan.

<u>PTO for proof.</u>

Formal proof of $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

STEP 1 We want to show $\forall n \in \mathbb{N}$ $2^n \geq n$, first.

Proof by induction : $p(n)$: $2^n \geq n$.

$p(1)$ $2^1 \geq 1$ ✓

want to show $\forall k \in \mathbb{N}$ $(2^k \geq k \Rightarrow 2^{k+1} \geq k+1)$

Induction hypothesis $2^k \geq k$.

$$2^{k+1} = 2^k \cdot 2 \geq 2k$$

$$k \geq 1 \Longrightarrow k + k \geq k+1$$

$$2k \geq k+1$$

$$2^{k+1} \geq k+1.$$ $p(k+1)$ is obtained.

By Thm of M. Induction: We showed that $\forall n \in \mathbb{N}$ $2^n \geq n$.

STEP 2 To show $\lim_{n\to\infty} \frac{1}{2^n} = 0$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon} > 0$

by Archimedean principle

$$\forall n \geq N, \quad |s_n - L| = \left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon. \#$$

since $2^n \geq n$.

Ex Show that $\lim_{n\to\infty} \frac{3n^2+1}{8n^2-7} = \frac{3}{8}$

Scrapwork:

$$\left|\frac{3n^2+1}{8n^2-7} - \frac{3}{8}\right| = \left|\frac{\cancel{24n^2}+8-\cancel{24n^2}+21}{8(8n^2-7)}\right|$$

$$= \frac{29}{8}\,\frac{1}{\underset{\overset{\vee}{0}}{(8n^2-7)}} = \frac{29}{64}\,\frac{1}{n^2-\frac{7}{8}} \overset{?}{\leq} \frac{29}{64}\cdot\frac{1}{n^2}$$

No

$$n^2 - \frac{7}{8} \geq n^2 \; ?$$

we can't do it

No

~~Another try~~

$$\frac{29}{64}\cdot\frac{1}{n^2-\frac{7}{8}} \leq \frac{29}{64}\,\frac{1}{\frac{1}{2}\cdot n^2} = \frac{29}{32}\,\frac{1}{n^2} < \varepsilon$$

need.

idea but not a proof.

need $n^2 - \frac{7}{8} \geq \frac{1}{2}n^2$

need $\frac{1}{2}n^2 \geq \frac{7}{8}$

need $n^2 \geq \frac{7}{4}$.

need. $n \geq 2$

PTO for the proof.

Actual Proof of $\lim_{n\to\infty} \frac{3n^2+1}{8n^2-7} = \frac{3}{8}$.

STEP 1. To show $\forall n \geq 2$ $\quad n^2 - \frac{7}{8} \geq \frac{1}{2}n^2$, first

$$n \geq 2$$
$$n^2 \geq 4 \geq \frac{7}{4}$$
$$\frac{1}{2}n^2 \geq \frac{7}{8}$$
$$n^2 \geq \frac{7}{8} + \frac{1}{2}n^2$$
$$n^2 - \frac{7}{8} \geq \frac{1}{2}n^2.$$

STEP 2 Let $\varepsilon > 0$ choose $N \in \mathbb{N}$ st.

$$N > \text{Max}\left(\sqrt{\frac{29}{32}\cdot\frac{1}{\varepsilon}}\,,\ 2\right)$$

so that $\frac{29}{32}\cdot\frac{1}{N^2} < \varepsilon$ and $N > 2$

$\forall n \geq N$

$$|s_n - L| = \left|\frac{3n^2+1}{8n^2-7} - \frac{3}{8}\right| = \left|\frac{\cancel{24n^2}+8-\cancel{24n^2}+21}{8(8n^2-7)}\right|$$

$$= \left|\frac{29}{64(n^2-\frac{7}{8})}\right| = \frac{29}{64}\,\frac{1}{\underbrace{(n^2-\tfrac{7}{8})}_{>0}} \underset{n \geq N > 2}{\leq} \frac{29}{64}\cdot\frac{1}{\frac{1}{2}n^2}$$

$n \geq 2$

$$= \frac{29}{32}\,\frac{1}{n^2} \leq \frac{29}{32}\,\frac{1}{N^2} < \varepsilon$$

\#

Max$(a_1, a_2, \ldots a_n)$ chooses the largest of $a_1, \ldots a_n$

min$(a_1, \ldots a_n)$ " " smallest of $a_1, \ldots a_n$.

Ex Show that $\lim\limits_{n\to\infty} n$ doesn't exist

or (n) is a divergent sequence.

$\lim\limits_{n\to\infty} s_n = L \iff \forall \varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\in\mathbb{N}\ (n\geq N \Rightarrow |s_n-L|<\varepsilon)$

s_n is convergent

$\iff \exists L\in\mathbb{R}\ \forall \varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\in\mathbb{N}\ (n\geq N \Rightarrow |s_n-L|<\varepsilon$

s_n is divergent

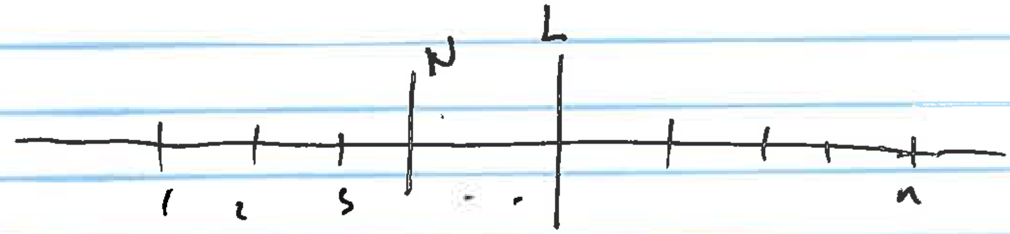
$$\iff \underbrace{\forall L\in\mathbb{R}\ \exists \varepsilon>0\ \forall N\in\mathbb{N}\ \exists n\in\mathbb{N}\ (n\geq N \text{ and } |s_n-L|\geq\varepsilon)}_{\text{Want}}$$

Scrapwork

I choose

$\underbrace{\forall L\in\mathbb{R}\ \exists \varepsilon=1\ \forall N\in\mathbb{R}}_{\text{given arbitrarily}}\ \exists n\geq N$

$|\underset{s_n}{n}-L| \underbrace{\geq 1=\varepsilon}_{\text{want}}$



If I choose n, $n>|L|+|N|+1$, then $\xrightarrow{}$ pro

(really large with respect to L & N)

given arbitrarily

since $|n| \geq n$ $\qquad |n| > |L| + |N| + 1$

I choose n

A working Idea ✓ $\qquad |n - L| \geq |n| - |L| \geq |N| + 1 > 1 = \varepsilon$

Now: Actual **Proof** $\forall L \in \mathbb{R}$ given arbitrarily.

I choose $\varepsilon = 1$

$\forall N \in \mathbb{N}$ given arbitrarily

I choose $\underset{\in \mathbb{N}}{n} > |L| + |N| + 1$, $\Rightarrow \underbrace{|n| \geq n > |L| + |N| + 1}$

then

$$|s_n - L| = |\underset{s_n}{n} - L| \underset{\text{reverse } \Delta\text{-inequality}}{\geq} \big|\underbrace{|n| - |L|}_{>0}\big| = |n| - |L|$$

$$\geq |N| + 1 \geq 1 = \varepsilon$$

Hence $\lim_{n \to \infty} n$ doesn't exist.