(3.2) Briefly

Field $(F, +, \cdot)$

$+ : F \times F \longrightarrow F$

$(a, b) \longmapsto a+b$

$\cdot : F \times F \longrightarrow F$

$(a, b) \longmapsto a \cdot b$

satisfying Axioms/ A1 – A5, M1 – M5, DL. } Textbook see pages 113–114

Examples of fields : $\underbrace{\mathbb{R}, \mathbb{Q}, \mathbb{C}}_{\text{usual operations}}$, $\mathbb{Z}_2 = \{0, 1\}$ mod 2 algebra

Examples not fields : $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^3$

fails : A5 ($\mathbb{N}$), M5 ($\mathbb{Z}$)

Ordered field Satisfying O.1 – 4 (p 114)

Ex $\mathbb{R}, \mathbb{Q}$

BUT $\mathbb{C}$ is not ordered

Completeness Axiom (will discuss later on in 3.3)

$\underbrace{\text{Field, Ordered, complete, contains } \mathbb{Q}.}$

$\mathbb{R}$ is the only one

Absolute value & Triangle inequality

Defn $|\cdot| : \mathbb{R} \to \mathbb{R}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Prop: $\forall x, y \in \mathbb{R}$, $\forall a \geq 0$

(a) $|x| \geq 0$

(b) $|x| \leq a \iff -a \leq x \leq a$

(c) $|xy| = |x||y|$

(d) $|x+y| \leq |x| + |y|$.

Proof of (d)

$-|x| \leq x \leq |x| \longleftarrow$ Why? **Lemma(1)**

Case 1 $x \geq 0$

$|x| = x$

$-|x| \leq 0 \leq x \leq |x|$

Case 2 $x < 0$

$|x| = -x$.

$x = -|x|$

$-|x| \leq x \leq 0 \leq |x|$

$$\begin{array}{rl} & -|x| \leq x \leq |x| \\ + & -|y| \leq y \leq |y| \\ \hline -(|x|+|y|) = & -|x|-|y| \leq x+y \leq |x|+|y| \end{array}$$

need $\longleftarrow$ **Lemma(2)** $a \leq b$, $c \leq d$.

(0.3) $a + c \leq b + c$

(0.3) $c + b \leq d + b$

(0.2) $a + c \leq b + d$

use (b) $\to$ $|x+y| \leq |x| + |y|$.

Order of the sections in our lectures.

Caution 3.1, 3.2, 4.1, 4.2, 4.3 / 3.3, 3.4, 4.4, +

omit 3.5, 3.6.

(*) Compactness: my notes, not 3.5

4.1

Defn A sequence is a function $s : \mathbb{N} \to \mathbb{R}$.

Notation: $s(n) = s_n$

$(s_1, s_2, s_3, \ldots s_n, \ldots)$, (s_n) ordered list

$\{s_n\}$ = range of (s_n) not ordered.

Ex (i) $\left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$

$\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} = \{\underbrace{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots}_{\text{may be rearranged}}\}$

(ii) $((-1)^n) = (-1, 1, -1, 1, -1, \ldots)$

$\{(-1)^n \mid n \in \mathbb{N}\} = \underbrace{\{1, -1\}}_{\text{range}}$

read the book 4.1.2 p 163.

<u>Def^n</u>: A sequence (s_n) is said to converge to a number $L \in \mathbb{R}$, if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ni \forall n \geq N \ |s_n - L| < \varepsilon.$$

$$(\Leftrightarrow) \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ni \forall n \in \mathbb{N} \ (n \geq N \Longrightarrow |s_n - L| < \varepsilon$$

A sequence (s_n) is said to diverge if there exists no $L \in \mathbb{R}$ to which s_n converges.

Notation $\lim\limits_{n \to \infty} s_n = L$, $s_n \to L$ when s_n converge to L.

Ex $\left(\frac{1}{n}\right)$, $\underline{\text{Prove}}$ $\frac{1}{n} \to 0$.

We will assume

<u>Archimedean Principle</u> ($\overset{3.3.}{\Longleftarrow}$ Completeness Axiom)

** $\forall x, y > 0 \ \exists n \in \mathbb{N} \quad nx > y$

($\Longleftrightarrow$ $\mathbb{N}$ is not a bounded set.)

<u>Ex</u> $\left(\frac{1}{n}\right)$: <u>Scrapwork:</u>

given

To choose for each given ε

<u>Want</u> $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \quad \forall n \in \mathbb{N}$

$$n \geq N \Longrightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$$

not a proof.

$$\left|\frac{1}{n} - 0\right| \leq \frac{1}{N} < \varepsilon$$ will choose $N > \frac{1}{\varepsilon} > 0$

want

Formal / Clean   Proof: Let $\varepsilon > 0$ be given. $(\forall)$

choose $N \in \mathbb{N}$, $N > \frac{1}{\varepsilon}$ $(\exists)$

by Archimedean Principle $x = 1$, $y = \frac{1}{\varepsilon}$

Hence $\frac{1}{N} < \varepsilon$.

$\forall n \geq N$, $n \in \mathbb{N}$

$$\Big|\underset{s_n}{\underset{\uparrow}{\tfrac{1}{n}}} - \underset{L}{\underset{\uparrow}{0}}\Big| = \Big|\frac{1}{n}\Big| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$