3.1 Continue Mathematical induction

A generalization of the standard Math. Induction:

<u>Thm:</u> Let $p(n)$ be a statement for $n \geq n_0$ for some given $n_0 \in \mathbb{N}$.

If

(i) $p(n_0)$ is true, and

(ii) $\forall k \geq n_0 \; (p(k) \Rightarrow p(k+1))$,

then $p(n)$ is true for $n \geq n_0$.

<u>Proof.</u> Let $q(\ell)$ be a statement for $\ell \geq 1$ defined by

$$q(\ell) = p(n_0 - 1 + \ell)$$

$$q(1) = p(n_0) \text{ true.}$$

$$\forall \ell \geq 1 \; (q(\ell) \Rightarrow q(\ell+1))$$

By Thm of induction, (the standard) $q(\ell)$ is true $\forall \ell \geq 1$.

$p(n)$ is true $\forall n \geq n_0$.

Exc 3.1.4

Prove $p(n)$ : $1^3+2^3+\dots+n^3 = \frac{1}{4}n^2(n+1)^2$ $\forall n \geq 1$

To prove by using Math. Induction by using $\circledast$, $\circledast\circledast$ below.

$\circledast$ $p(1)$ : $1^3 \overset{?}{=} \frac{1}{4} 1^2 \cdot (1+1)^2 = 1$. ✓ True.

Assume $p(k)$ to prove $p(k+1)$.

$p(k)$ : $1^3+2^3+\dots+k^3 = \frac{1}{4}k^2(k+1)^2$. assumed Induction Hypoth.

$$1^3+2^3+\dots+k^3 = \frac{1}{4}k^2(k+1)^2$$

$$1+2^3+\dots+k^3+(k+1)^3 = \frac{1}{4}k^2(k+1)^2+(k+1)^3.$$

$$= (k+1)^2\left[\frac{1}{4}k^2+k+1\right].$$

$$= (k+1)^2 \frac{1}{4}\left[\underbrace{k^2+4k+4}\right].$$

$$= (k+1)^2 \cdot \frac{1}{4}(k+2)^2.$$

$$\underbrace{1+2^3+\dots+(k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2}_{\text{This is } p(k+1)}$$

Done.

$\circledast\circledast$ We proved $\forall k\,(p(k) \Rightarrow p(k+1))$.

3.1.7 Prove by induction

$p(n): \quad 1+r+r^2+\cdots+r^n = \dfrac{1-r^{n+1}}{1-r}$ th, $r \neq 1$.

Proof by induction

$p(1) \qquad 1+r \overset{?}{=} \dfrac{1-r^2}{1-r} = \dfrac{(1-r)(1+r)}{1-r} = 1+r$ when $r \neq 1$.

$p(1)$ is true.

To prove $\forall k \in \mathbb{N} \left( p(k) \Longrightarrow p(k+1) \right)$

Assume Induction Hypothesis

$$1+r+\cdots+r^k = \frac{1-r^{k+1}}{1-r}.$$

$$1+r+\cdots+r^k+r^{k+1} = \frac{1-r^{k+1}}{1-r} + r^{k+1}$$

$$= \frac{(1-r^{k+1}) + (1-r)\,r^{k+1}}{1-r}$$

$$= \frac{1-\cancel{r^{k+1}} + \cancel{r^{k+1}} - r^{k+2}}{1-r}$$

$$= \frac{1-r^{k+2}}{1-r}$$

We obtained $p(k+1): \; 1+r+\cdots+r^{k+1} = \dfrac{1-r^{k+2}}{1-r}$.

By Thm of induction $\forall n \in \mathbb{N} \; \forall r \neq 1$, $p(n)$ is true.

Exc. 3.1.23(b) $p(n): n^2 \leq 2^n \quad \forall n \geq ?$

$n=1 \quad 1^2 \leq 2^1$ ✓

$n=2 \quad 2^2 \leq 2^2$ ✓

$n=3 \quad 9=3^2 \not\leq 2^3=8 \quad p(3)$ is false.

$n=4 \quad 4^2 \leq 2^4$ ✓

Claim: $\forall n \geq 4 \quad n^2 \leq 2^n \quad (n \in \mathbb{N}) \quad p(n)$.

This is NO PROOF

Try / Scrapwork:

Will assume $k^2 \leq 2^k$

Want $(k+1)^2 \leq 2^{k+1}$

$$2^{k+1} = 2 \cdot 2^k \geq 2k^2 \overset{\text{want}}{\geq} (k+1)^2$$

can I get this?

Is it true?

? $2k^2 \geq k^2 + 2k + 1$

? $k^2 - 2k - 1 \geq 0$

? $(k-1)^2 - 2 \geq 0$

? $(k-1)^2 \geq 2$ True if $k \geq 4$.

good but not a proof

Now we write the actual proof, and lose this page.

Exc. 3.1.23(b)

Actual proof:

Let $p(n)$: $n^2 \leq 2^n$ $\forall n \in \mathbb{N}$ $n \geq 4$.

$p(4)$ $16 = 4^2 \leq 2^4 = 16$ is true.

To prove $\forall k \geq 4$ $(p(k) \Rightarrow p(k+1))$.

Assume induction hypothesis.

$$k^2 \leq 2^k, \text{ and}$$
$$k \geq 4$$
$$k-1 \geq 3$$
$$(k-1)^2 \geq 9 \geq 2.$$
$$(k-1)^2 - 2 \geq 0$$
$$k^2 - 2k + 1 - 2 \geq 0$$
$$k^2 - 2k - 1 \geq 0$$
$$\underbrace{2k^2 \geq k^2 + 2k + 1 = (k+1)^2}$$

$$2^{k+1} = 2 \cdot 2^k \geq 2k^2 \geq (k+1)^2$$

used $2^k \geq k^2$ $(p(k))$

obtained $2^{k+1} \geq (k+1)^2$ i.e. $p(k+1)$.

- $\forall k \geq 4 \, (p(k) \Rightarrow p(k+1))$.
- $p(4)$ is true

Thm of Math. Induction $\Rightarrow$ $p(n)$ is true for all $n \geq 4$.

What is wrong with this proof?

Ex $p(n)$: $n^2 + 9n + 1$ is even, TRUE?? Actually NOT

$$\boxed{\forall k \left( p(k) \Rightarrow p(k+1) \right) \text{ is True.}}$$

A correct proof of "wrong $\Rightarrow$ wrong"

$p(k+1)$: $(k+1)^2 + 9(k+1) + 1$ is even

$$(k+1)^2 + 9(k+1) + 1 = k^2 + 2k + 1 + 9k + 9 + 1$$
$$= \underbrace{k^2 + 9k + 1}_{\text{even}} + \underbrace{2k + 10}_{\text{even}} \text{ is even.}$$

induction hypothesis $p(k)$

What is wrong with this then?

$p(1)$ 11 is even? X NO

$p(2)$ 23 is even? X NO

We Didn't check $P(1), P(2), \ldots$ They are all WRONG.

Lesson: Must check $P(1)$ AND

Must check $\forall k (P(k) \Rightarrow P(k+1))$, BOTH

if either missing, then you do not have a proof