Thm: $\mathbb{R}$ is uncountable

Sketch the idea:
Cantor's Diagonal argument

- Will show open interval $(0,1)^{\subseteq \mathbb{R}}$ is uncountable
$\Longrightarrow$ $\mathbb{R}$ is uncountable (Thm(i) earlier)

Recall

We will prove this later

$\forall x \in (0,1)$ has a decimal expansion $\rightarrow$ $\frac{1}{3} = 0.\bar{3}$, $\frac{1}{8} = 0.125$
$\pi = 3.14159\ldots$

Not unique: $0.\bar{9} = 1 = 1.\bar{0}$

Similarly $0.1357 = 0.13569\bar{9} \in (0,1)$

Proof by contradiction:
Suppose $(0,1)$ is countable, and $\exists$ bijection $f : \mathbb{N} \to (0,1) (\subseteq \mathbb{R})$
one-to-one and onto

$x_1 = f(1) = 0.\,a_{11}\,a_{12}\,a_{13}\,a_{14} \ldots$ decimal expansion
$x_2 = f(2) = 0.\,a_{21}\,a_{22}\,a_{23}\,a_{24} \ldots$
$a_{ij} \in \{0,1,\ldots 9\}$ $x_3 = f(3) = 0.\,a_{31}\,a_{32}\,a_{33}\,a_{34} \ldots$
$x_4 = f(4) = 0.\,a_{41}\,a_{42}\,a_{43}\,a_{44} \ldots$
$\vdots$ Look at the diagonal

Countable list of all $f(n)^{n \in \mathbb{N}}$, supposedly all of the interval $(0,1)$.

Take $A = 0.\,b_1\,b_2\,b_3\,b_4 \ldots$ where $b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$

$\forall n.$ $A \neq f(n)$ since $b_n \neq a_{nn}$; $A \in (0,1)$, f is NOT onto X

2

Exec. 2.4. a e f c $\quad S, T \subseteq \mathbb{R}$ Find bijections between pairs.

(a) $S = [0,1] \xrightarrow[\text{bij}]{f} T = [1,4]$

$$f(x) = 3x + 1$$

(f) $S = (0,1) \xrightarrow[\text{bij}]{g} T = \mathbb{R}.$
interval

$$g(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$$

(e) $S = (0,1) \xrightarrow[\text{bij}]{h} T = (0, \infty)$

$$h(x) = \frac{1}{x} - 1$$

(c) $S = [0,1] \xrightarrow{\ell} T = [0,1)$

Take $\ell(x) = \begin{cases} x & \text{if } x \neq \frac{1}{n} \text{ th } n \in \mathbb{N}. \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \end{cases}$

since there is an obvious bijection between

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\} \longrightarrow \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$$

$$\frac{1}{n} \longmapsto \frac{1}{n+1}$$

Chap III

3.1 $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ natural numbers.

AXIOM of WELL-ORDERING of $\mathbb{N}$

For any non-empty subset S of $\mathbb{N}$, there exist a smallest element m of S.

Equivalently

$$\forall S \subseteq \mathbb{N} \left(S \neq \phi \Rightarrow \exists m \in S \; \forall n \in S \; (m \leq n) \right)$$

Caution This is false in $\mathbb{Z}$, $\mathbb{R}$, $(0, \infty)$, $[0, \infty)$

Theorem: (Principle of Mathematical Induction)

Let $p(n)$ denote a stament for each $n \in \mathbb{N}$.

If (i) $P(1)$ is true, and
(ii) $\forall k \in \mathbb{N} \; (p(k) \Rightarrow p(k+1))$, then

$\forall n \in \mathbb{N} \; p(n)$ is true.

Remark Axiom of Well ordering of $\mathbb{N}$ $\Longrightarrow$ Principle of mathematical induction.

$\Longrightarrow$ All examples proven by using Principle of mathematical induction

This is a deductive procedure

Principle of Mathematical Induction. Proof by Contradiction

<u>Proof</u> Suppose $p(n_0)$ is false for some $n_0 \in \mathbb{N}$.

Let $S = \{ n \in \mathbb{N} \mid p(n) \text{ is false} \}$.

$S \neq \phi$, $n_0 \in S$

Axiom of Well-ordering $\Rightarrow$ S has a smallest element

Let's call $m \underset{\wedge}{=}$ smallest element of S.

$m \in S$

$p(m)$ is false. ⊛

$m \neq 1$ since $p(1)$ is true.

$m > 1$, $m \in \mathbb{N}$

$m-1 \in \mathbb{N}$, $m-1 \notin S$ (m was the smallest elt. of S)

$p(m-1)$ is true. ⊛⊛

$\forall k \in \mathbb{N}, \; (p(k) \Rightarrow p(k+1))$ — Hypothesis

particularly $\underbrace{p(m-1)}_{\text{True ⊛⊛}} \Rightarrow \underbrace{p(m)}_{\text{false ⊛}}$. True can't imply false

Contradiction.

The supposition "$p(n_0)$ is false" can't happen.

$\forall n \; p(n)$ is true. #