2.3 To conclude

Defn Let $f : A \to B$ be bijective (1-1 & onto)
One defines a function

$$f^{-1} : B \to A \text{ by}$$

$$b = f(a) \iff a = f^{-1}(b)$$

Remark: (Inverse) image or preimage is defined for all functions, regardless of whether the function is invertible or not.
When a function is invertible pre-image & inverse function match:

Obs (i) $f^{-1}(\{b\}) \neq \phi \ \forall b \in B \qquad (f : A \to B)$

$$\Updownarrow$$

f is surjective

(ii) $f^{-1}(\{b\})$ has at most one element, $\forall b \in B$

$$\Updownarrow$$

f is injective

(iii) $f^{-1}(\{b\})$ has exactly one element, $\forall b \in B$.

$$\Updownarrow$$

f is bijective $\iff$ f is invertible

Compare Notation:

$f^{-1}(b) = a$	inverse function
$f^{-1}(\{b\}) = \{a\}$	preimage

Prop $f: A \to B$ is bijective, then

(i) $f^{-1}: B \to A$ is bijective

(ii) $f^{-1} \circ f = Id_A$ $\qquad f \circ f^{-1} = Id_B$.

HW to read

Prop Let $f: A \to B$

$g: B \to C$ be both bijective.

Then (i) $g \circ f : A \to C$ is bijective

(ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Example:

Caution: $(g \circ f)^{-1}$ exists doesn't mean both f & g are invertible

$$\left.\begin{array}{ll} f(x) = (x, 0) & : \mathbb{R} \to \mathbb{R}^2 \\ g(x, y) = x & : \mathbb{R}^2 \to \mathbb{R} \end{array}\right\}$$ neither is invertible

$$(g \circ f)(x) = g(x, 0) = x = Id_{\mathbb{R}}(x)$$

$(g \circ f)^{-1}$ exists.

Responsible for
2.4 (Only the Lecture material.)

<u>Defn</u> Two sets S and T are called equinumerous if there is a bijection between them.

We will discuss:

- finite set } countable
- denumerable } countable; } infinite
- uncountable } infinite

<u>Defn</u> A set S is called finite if either
(i) $S = \emptyset$, or
(ii) $\exists$ bijection $f: \{1, 2, 3, \ldots n\} \longrightarrow S$

$\longrightarrow$ A set is called infinite if it is not finite.

<u>Informally</u>: The sets one can count & (theoretically) finish counting, are finite.

Ex $\{a, b, c\} \longleftrightarrow \{1, 2, 3\}$.

$\{n \in \mathbb{Z} \mid |n| \leq 10^{10^{10}}\}$ is a finite set

$\{$electrons in earth$\}$ is a finite set

Not finite = infinite

<u>ex.</u> $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, (0, 1] \subseteq \mathbb{R}$.

<u>Defn</u> A set S is called denumerable if there exists a bijection

$$f : \mathbb{N} \longrightarrow S.$$

A set S is called countable if it is either finite or denumerable.

A set S is called uncountable if it is not countable (i.e. not finite & not denumerable.)

Ex (i) $\mathbb{N}$ denumerable.

(ii) $\mathbb{Z}$ denumerable:

Find $f : \mathbb{N} \longrightarrow \mathbb{Z}$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even.} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

(iii) $\mathbb{Q}$ denumerable

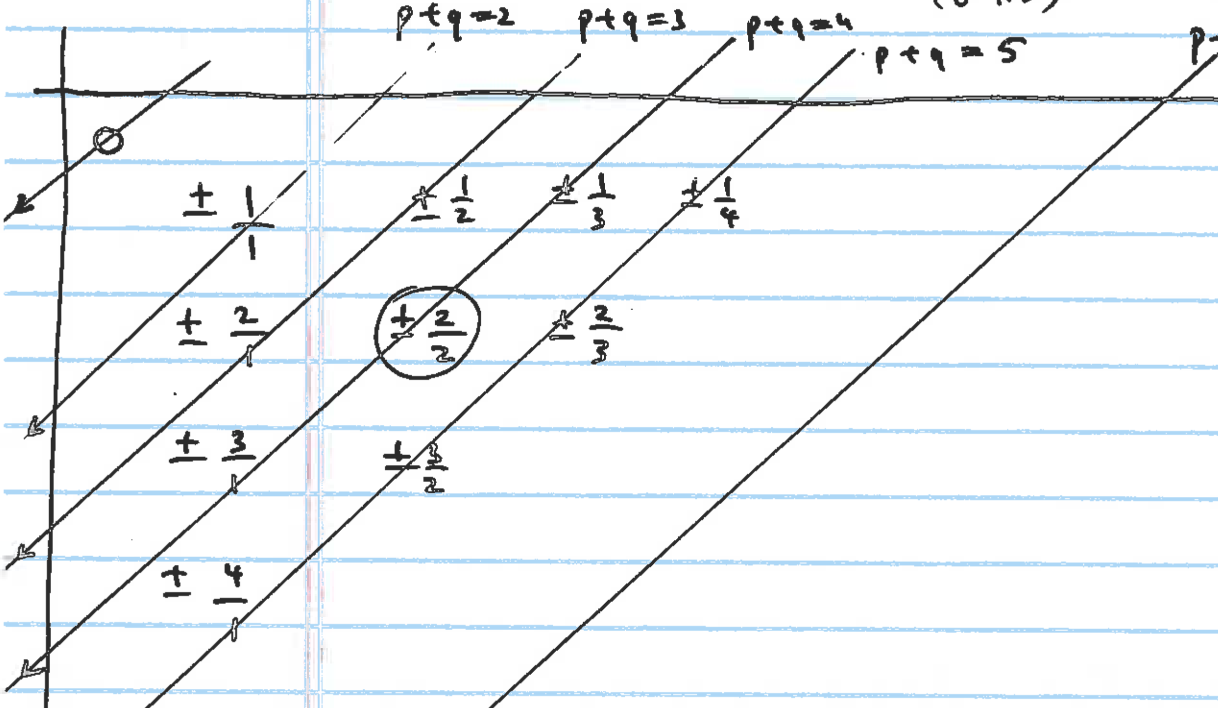
(i) informal

(ii) formal

Informal way to count/list $\mathbb{Q}$. (5)

List all $\pm\frac{p}{q}$, $p, q \in \mathbb{N}$, following 0.

(only 2 #s) (4 #s) (6 #s) (8 #s) Total $2N-2$ such



omit $\pm\frac{2}{2} = \pm 1$ used earlier

List in order

$0, 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1}, \frac{1}{4}, -\frac{1}{4}$.

Discussion of any number $\pm\frac{p}{q}$ will be reached by an estimated number of steps:

The total number of $\pm\frac{p}{q}$ above the diagonal $p+q=N$ inclusively (without omitting any for repetition) is

$$1+2+4+6+8+\cdots 2N-2 = 1+2(1+2+3+\cdots+N-1)$$
$$= 1+2\cdot\frac{N(N-1)}{2} = 1+N^2-N$$

Hence any rational number of the form $\pm\frac{p}{q}$ will be counted for before reaching N^2-N+2, ($N=p+q$.)

Thm:

(i) S is countable & $T \subseteq S \implies T$ is countable

(ii) S & T are countable $\implies S \times T$ countable

(iii) Union of countably many countable sets is countable.

Discuss $\mathbb{Q}$ again:

Method I

For each $q \in \mathbb{Z},\ q \neq 0$,

define $A_q = \{ \frac{1}{q}, \frac{2}{q}, \frac{3}{q}, \frac{4}{q}, \ldots, \frac{n}{q}, \ldots \} \xleftrightarrow{\text{bijective}} \mathbb{N}$.

Thm (iii).

$$\mathbb{Q} = \{0\} \cup \bigcup_{\substack{q=-\infty \\ q \in \mathbb{Z}-\{0\}}}^{\infty} A_q$$

is countable since each A_q is countable & $q \in \mathbb{Z} - \{0\}$ is countable.

Method II

$$S = \{ (p, q, \sigma) \mid p \in \mathbb{N},\ q \in \mathbb{N},\ \sigma = 1 \text{ or } -1 \}$$
$$= \mathbb{N} \times \mathbb{N} \times \{-1, 1\} \text{ is countable by Thm (ii)}$$

$S' = \{ (p, q, \sigma) \mid p \text{ \& } q \text{ are relatively prime} \} \subseteq S$

S' is countable by Thm (i)

$$S' \xrightarrow{\text{bijection}} \mathbb{Q} - \{0\} \quad \text{countable}$$
$$(p, q, \sigma) \longmapsto \sigma \cdot \frac{p}{q} \qquad (\sigma = 1 \text{ or } -1)$$

$\mathbb{Q} = (\mathbb{Q} - \{0\}) \cup \{0\}$ is countable by thm (iii)

Next time we will show (Friday)

Thm: $\mathbb{R}$ is uncountable.

Prop: Irrationals $I = \mathbb{R} \setminus \mathbb{Q}$ is uncountable.

Proof Suppose $\mathbb{R} \setminus \mathbb{Q} = I$ were countable.

Then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would be countable by Thm(iii). (union of 2 countable sets.)

However $\mathbb{R}$ is not countable.

Contradiction.

Hence the supposition of $\mathbb{R} \setminus \mathbb{Q}$ is countable is false.

$I = \mathbb{R} \setminus \mathbb{Q}$ is uncountable. #