Sept 10, 2018

1

p 70 Thm. 2.3.16 (c)

Given $f: A \to B$

To prove $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$, $\forall C_1, C_2 \subseteq A$.

Ex It is possible to have $f(C_1 \cap C_2) \neq f(C_1) \cap f(C_2)$.

Take $f(x) = \sin x$ $C_1 = \{\frac{\pi}{2}\}$ $C_2 = \{\frac{5\pi}{2}\}$.

$$C_1 \cap C_2 = \phi.$$

$$f(C_1) = \{1\} = f(C_2)$$

$$\left.\begin{array}{l} f(C_1) \cap f(C_2) = \{1\} \\ f(C_1 \cap C_2) = f(\phi) = \phi. \end{array}\right] \neq.$$

Proof of (c) If $C_1 \cap C_2 = \phi$, $f(\phi) = \phi \subseteq f(C_1) \cap f(C_2)$

If $C_1 \cap C_2 \neq \phi$,

then $f(C_1 \cap C_2) \neq \phi$:

Let $b \in f(C_1 \cap C_2)$ be any element.

$$b = f(a) \text{ for some } a \in C_1 \cap C_2$$

$$a \in C_1, \text{ and } a \in C_2$$

$$f(a) \in f(C_1) \text{ and } f(a) \in f(C_2)$$

$$b = f(a) \in f(C_1) \cap f(C_2).$$

We showed

$$\forall b \left(b \in f(C_1 \cap C_2) \Rightarrow b \in f(C_1) \cap f(C_2). \right)$$

Thm 2.3.16 (b) $f(f^{-1}(D)) \subseteq D$.

Homework to prove

Ex $f: \mathbb{R} \to \mathbb{R}$. $f(x) = x^2$. $D = \mathbb{R}$. } showing $f(f^{-1}(D)) \neq D$ is possible

$$f(f^{-1}(D)) = f(\underbrace{f^{-1}(\mathbb{R})}_{\mathbb{R}}) = f(\mathbb{R}) = [0, \infty) \neq \mathbb{R} = D.$$

Thm 2.3.18 $f: A \to B$.

stronger than the book.

(a) f is injective $\iff$ $\forall C \subseteq A$, $f^{-1}(f(C)) = C$

(b) f is surjective $\iff$ $\forall D \subseteq B$ $f(f^{-1}(D)) = D$

(c) f is injective $\iff$ $\forall C_1, C_2 \subseteq A$ $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$

(b) is HW

Proof of: (a) ($\Rightarrow$:)

Assumption f is injective

To prove $\forall C \subseteq A \quad f^{-1}(f(C)) = C$

we need $\left[\begin{array}{l} f^{-1}(f(C)) \subseteq C \\ C \subseteq f^{-1}(f(C)) \end{array} \right.$ · Next to do. · Done earlier

True for all functions Thm 2.3.16(a)

To prove $f^{-1}(f(C)) \subseteq C$

If $f^{-1}(f(C)) = \phi$, then $\phi \subseteq C$. ✓

If $f^{-1}(f(C)) \neq \phi$, Let

$a \in f^{-1}(f(C))$ be any element.

$f(a) \in f(C)$

$f(a) = f(c)$ for some $c \in C$

We know f is injective.

$a = c \in C$

$a \in C$

Recall:

Def^n: $a \in f^{-1}(D) \Leftrightarrow f(a) \in D$

Def^n: $b \in f(C) \Leftrightarrow b = f(c)$ for some $c \in C$.

$\forall a \in A \ (\ a \in f^{-1}(f(C)) \Rightarrow a \in C\ )$

$f^{-1}(f(C)) \subseteq C$. #

Not in the book ($\Leftarrow$:) Assume $\forall C_1 \subseteq A \quad f^{-1}(f(C_1)) = C_1$

To prove f is injective $\forall a, b \in A (f(a) = f(b) \Rightarrow a = b)$

Let $a, b \in A$ s.t. $f(a) = f(b) = k$.

Take $C = \{a\}$

$f(C) = \{k\}$

$\{a\} = C = f^{-1}(f(C)) = f^{-1}(\{k\}) \supseteq \{a, b\}$

$\{a, b\} \subseteq \{a\}$

$a = b$.

Composition of functions

<u>Defn</u> Let $f: A \to B$ &
$g: B \to C$, then one
defines $g \circ f : A \to C$ by

$$(g \circ f)(a) = g(f(a)).$$

2.3.20
p 72

<u>Thm:</u> Let $f: A \to B$
$g: B \to C$

(a) Both f, g surjective $\Rightarrow$ $g \circ f$ is surjective
(b) Both f, g injective $\Rightarrow$ $g \circ f$ is injective
(c) " f, g bijective $\Rightarrow$ $g \circ f$ is bijective.

(b) HW
(a & b $\Rightarrow$ c)

<u>Proof</u> (a) $g \circ f: A \to C$.
Let $c \in C$ be any elt.
$g: B \to C$ onto/surjective
$\exists b \in B \quad g(b) = c$.
$f: A \to B$ onto/surjective
$\exists a \in A \quad f(a) = b$.
$c = g(b) = g(f(a)) = (g \circ f)(a)$
$g \circ f$ is surjective (onto C).

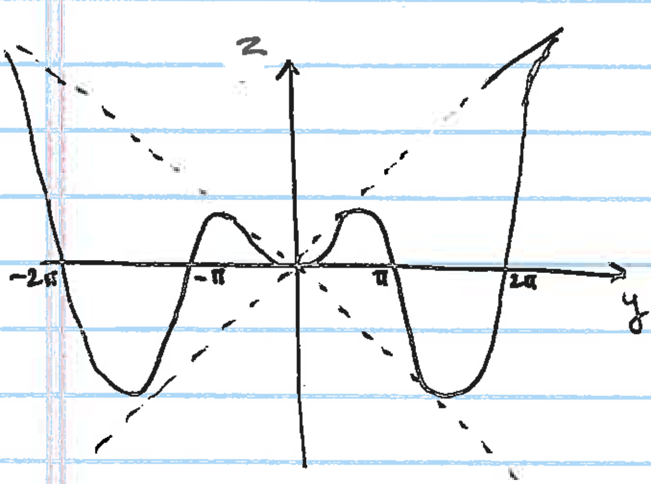
<u>Converse:</u> $g \circ f$ surjective $\overset{?}{\Longrightarrow}$ g & f are both surjective.?
Question

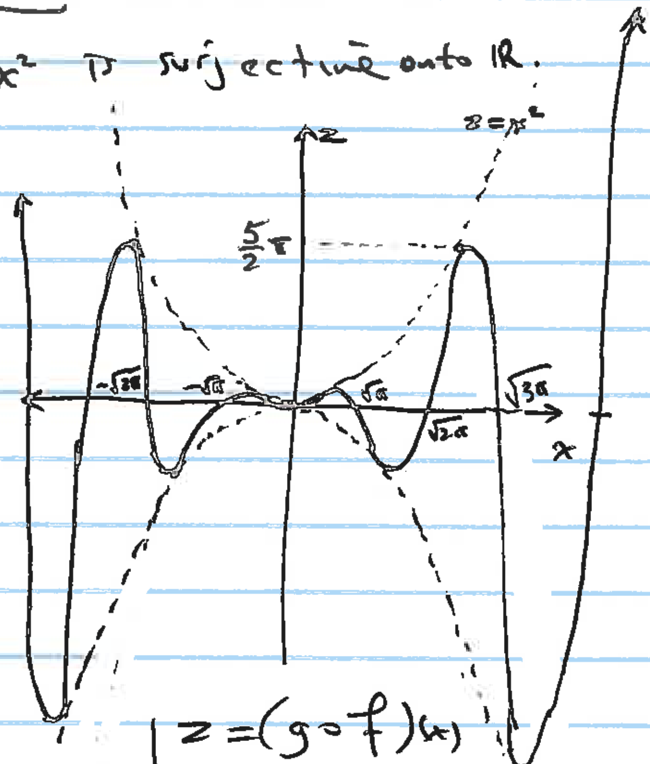
<u>Answer</u> No

Counterexample: $g \circ f$ & g surjective but not f:

$$\underbrace{\mathbb{R} \xrightarrow{f(x)=x^2} \mathbb{R}}_{\text{not surjective}} \; \underbrace{\xrightarrow{g(y)=y\sin y} \mathbb{R}}_{\text{surjective}}$$

$z = (g \circ f)(x) = x^2 \sin x^2$ is surjective onto $\mathbb{R}$.





$f(\mathbb{R}) = [0,\infty)$	$z = g(y) = y \sin y$ $g(\mathbb{R}) = \mathbb{R}$. $g([0,\infty)) = \mathbb{R}$	$(g \circ f)(\mathbb{R})$ $= g([0,\infty)) = \mathbb{R}$	$z = (g \circ f)(x)$ $= x^2 \sin x^2$

TRUE HOWEVER $\quad g \circ f$ surjective $\Longrightarrow$ g surjective

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$g \circ f$ surjective

$(g \circ f)(A) = C$

$g(f(A)) = C$

$f(A) \subseteq B$

$C = g(f(A)) \subseteq g(B)$

$C \subseteq g(B)$

$g(B) \subseteq C \leftarrow$ always true for any function

g is surjective $\Leftarrow$ $\quad C = g(B)$