2.2

Defn Let A & B be sets. Every subset $R \subseteq A \times B$ is called a relation between A & B.
If $R \subseteq A \times A$, then R is called a relation on A.

Notation $(a,b) \in R \iff aRb$

Caution
$R \neq \mathbb{R}$

Ex $\{(x,y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$ is a relation on $\mathbb{R}$
$\{(x,y) \mid x \leq y\} \subseteq \mathbb{R} \times \mathbb{R}$ is a relation on $\mathbb{R}$

2.3

Defn Let A and B be sets.
A function f from A to B is a relation $f \subseteq A \times B$ satisfying

(i) $\forall a \in A \; \exists b \in B$ s.t. $(a,b) \in f$

(ii) $\forall a \in A \; \forall b, c \in B$
$(a,b) \in f$ & $(a,c) \in f \implies b = c$.

A = domain of f
B = codomain of f

Range of f = rng f = $\{b \in B \mid \exists a \in A \; (a,b) \in f\}$

Notation $\begin{cases} f : A \longrightarrow B \\ f(a) = b \iff (a,b) \in f. \end{cases}$

Compare: $f(x)=x^2$ by itself, not giving all of the information

$$\left.\begin{array}{l} f_1(x) = x^2 : \mathbb{R} \longrightarrow \mathbb{R} \\ f_2(x) = x^2 : \mathbb{R} \longrightarrow [0,\infty) \\ f_3(x) = x^2 : [0,\infty) \longrightarrow [0,\infty) \end{array}\right\} \text{ not the same function.}$$

__Defn__ Let $f: A \to B$ be a function

(i) f is called onto if $\operatorname{rng} f = B$
$(\iff \forall b \in B\ \exists a \in A \ni f(a)=b)$

onto = surjective

(ii) f is called one-to-one (= injective) if

$$\forall x_1, x_2 \in A \ (f(x_1)=f(x_2) \Rightarrow x_1 = x_2)$$

$$(\iff \forall x_1, x_2 \in A \ (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$$

(iii) f is called bijective (= one-to-one correspondence) if f is both injective & surjective.

When/how do I know a relation is a function?
Vertical line test:
If $f: A \to B$
$A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$,
Every vertical line $x=a$ intersects the graph of f (in $\mathbb{R}^2$) at at most one point.

If $x = a \in A$, then there must be exactly one pt of intersection.

If $x = a \notin A$, then there doesn't exist any pt of intersection.

Def Let $f : A \to B$ be a function.

(i) For any $C \subseteq A$, the image (or direct image) of C is defined by

$$f(C) = \{ f(x) \mid x \in C \}$$

(ii) For any $D \subseteq B$, the pre-image of D is defined by

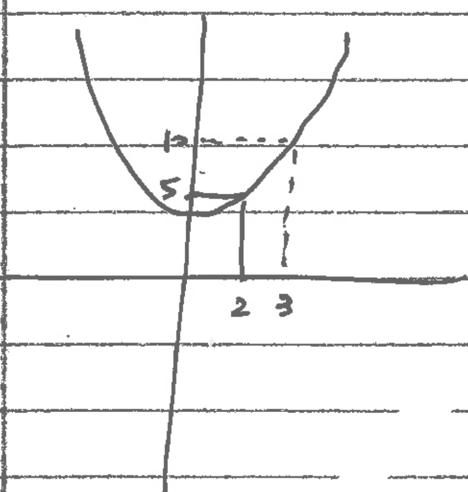
$$f^{-1}(D) = \{ x \in A \mid f(x) \in D \}$$

CAUTION This is more generalized than the notion of an inverse function.

The pre-image is defined for any function f regardless of f is invertible or not.

Ex $f(x) = x^2+1 : \mathbb{R} \to \mathbb{R}$.

$f([2,3]) = [5,10]$.



$f((-1,2]) = [1,5]$

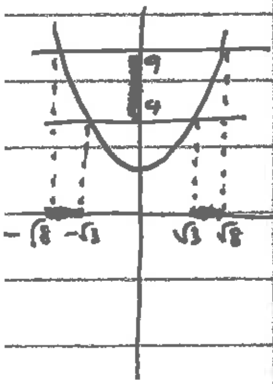
$f(\mathbb{R}) = [1,\infty)$

$f^{-1}([0,1]) = \{0\}$

$0 \le x^2+1 \le 1$
$-1 \le x^2 \le 0$
$x = 0$

$f^{-1}([4,9]) = [\sqrt{3},\sqrt{8}] \cup [-\sqrt{8},-\sqrt{3}]$.

$\|$



$\{x \in \mathbb{R} \mid f(x) = x^2+1 \in [4,9]\}$

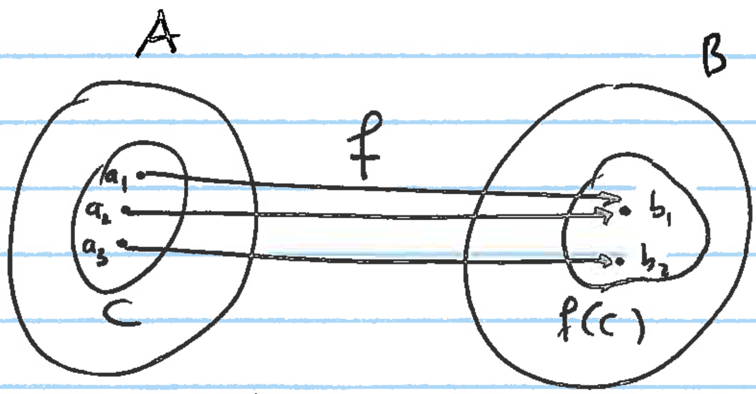
$4 \le x^2+1 \le 9$

$3 \le x^2 \le 8$

$f^{-1}((-\infty,0)) = \phi$.

i.e. $\exists$ no x s.t. $x^2 < 0$.

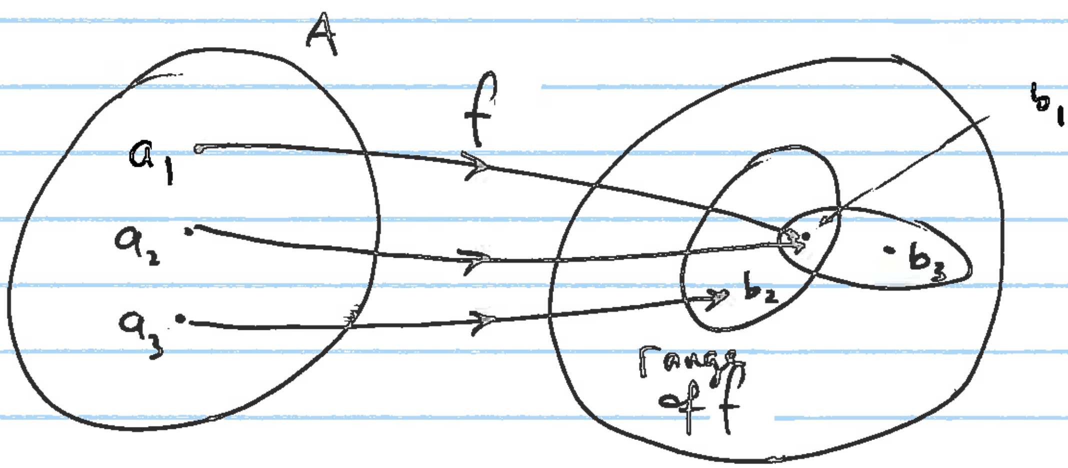
Direct image



$$C = \{a_1, a_2, a_3\}$$

$$f(C) = \{f(a_1), f(a_2), f(a_3)\} = \{b_1, b_2\}$$

Pre-Image



$$f(a_1) = f(a_2) = b_1$$

$$b_3 \notin \text{range of } f$$

$$f^{-1}(\{b_1\}) = \{a_1, a_2\}$$

$$f^{-1}(\{b_3\}) = \emptyset.$$

Thm 2.3.16 p 70 $f: A \to B$ (Read from the book)

(a) $\forall C \subseteq A, \quad C \subseteq f^{-1}[f(C)]$.

Caution (Ex. $C \neq f^{-1}(f(C))$ is possible)

$f(x) = x^2 \qquad C = \{1\}$

$f(C) = \{1\}$

$f^{-1}(\{1\}) = \{-1, 1\} \neq C$.

Want to Prove (a) $\forall x \in A \ (x \in C \Rightarrow x \in f^{-1}(f(C)))$

Proof: If $C = \emptyset$, nothing to prove.

If $C \neq \emptyset$, let $a \in C$ be an arbitrary element.

$f(a) \in f(C) = D$. we renamed it

$f(a) \in D$.

(Defn $\Rightarrow$) $f^{-1}(D) = \{x \in A \mid f(x) \in D\}$

$a \in \underbrace{f^{-1}(D) = f^{-1}(f(C))}_{\text{we chose } D = f(C)}$

Hence $\forall x \quad x \in C \Rightarrow x \in f^{-1}(f(C))$

$C \subseteq f^{-1}(f(C))$.

Thm 2.3.16 (f) $\quad f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

Proof: $\forall x \in A$

$$\begin{aligned} x \in f^{-1}(D_1 \cup D_2) &\iff f(x) \in D_1 \cup D_2. \\ &\iff f(x) \in D_1 \text{ or } f(x) \in D_2 \\ &\iff x \in f^{-1}(D_1) \text{ or } x \in f^{-1}(D_2) \\ &\iff x \in f^{-1}(D_1) \cup f^{-1}(D_2). \end{aligned}$$

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