Aug 31, 2018

2.1 Sets

→ Certain collections of objects characterized by some defining properties are called set.

Not every collection is a set.

→ The objects in a set are called the elements of the set.

$x \in A$ means x is an element of the set A.

→ $\{\ldots\}$ are used to show sets.

Ex. $2 \in \{2, 3, 5\}$ $\quad$ $6 \notin \{2, 3, 5\}$

$\{\ \} = \varnothing$ empty set

$$\underbrace{\{x \in A \mid p(x)\} \text{ or } \{x \in A : p(x)\}}$$

The collection of elements in A satisfying the $p(x)$.

$2 \in \{2\}$

$\{2\} \notin \{2\}$

$2 \notin 2$

$\{2\} \in \{\{2\}\}$

$\varnothing = \{\ \}$ $\quad$ $\{\varnothing\} = \{\{\ \}\}$

Defn Let A and B be sets.

(i) $A \subseteq B$ (A is a subset of B),
if every element x of A, x is also an element of B.

$$A \subseteq B \text{ if } \forall x\,(x \in A \Rightarrow x \in B)$$

In every definition "if" means "iff". BUT

Nowhere else this happens. "if" and "iff" in a statement are different.

Defn (ii) $A = B$ if $(A \subseteq B$ and $B \subseteq A)$

(iii)

$A \subsetneq B$ if $(A \subseteq B$ and $A \neq B)$

(iv) $A \nsubseteq B$ if (not $A \subseteq B$).

$$\Updownarrow$$

$$\text{not } \forall x\,(x \in A \Rightarrow x \in B)$$

$$\Updownarrow$$

$$\exists\, x \in A \text{ and } x \notin B.$$

Defn $\mathbb{N} = \{1, 2, 3, 4, \dots\dots\}$

natural numbers

$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

whole numbers.

$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ q \neq 0 \right\}$

$\mathbb{R}$ = real numbers

Defn $[a,b]$, $(a,b]$, (a,b), $[a,b)$

$(-\infty, a]$, $(-\infty, a)$, (b, ∞), $[b, \infty)$

$[a,b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$

$(a,b) = \{ x \in \mathbb{R} \mid a < x < b \}$

$\vdots$

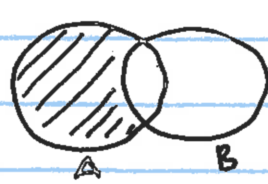
Defn Let A & B be sets

$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$

$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

$$A \sim B = A \setminus B = A - B = \{ x \in A \mid x \notin B \}$$

Venn diagrams are for Visualization

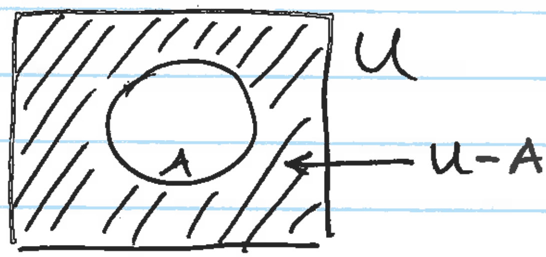


Venn diagrams are <u>not</u> proofs.

<u>Complement</u> needs a specific universal set U.

For a given universal set U, $A \subseteq U$, one defines the complement of A in U to be

$$U \setminus A = \{x \in U \mid x \notin A\}$$



"Everything" is <u>not</u> a set

Russel's Paradox:

If one takes $U = \{A \mid \underbrace{A \notin A}_{\circledast}\}$, there is a paradox:

If $U \in U$, then U satisfies the condition $\circledast$, i.e. $U \notin U$.
If $U \notin U$, then U doesn't satisfy the condition $\circledast$, i.e. $U \in U$.

Hence $U \in U \iff U \notin U$ which is a paradox, a true statement can't be equivalent to a false statement. What is the problem? U is not a set.

We will Do Basic proofs in sets:

Thm: For any set A, $\phi \subseteq A$

Proof $\underbrace{\forall x\,(\underbrace{x \in \phi}_{\text{false}} \Rightarrow x \in A)}_{\text{True.}}$

HW Practice 2.1.11

read Prop 2.1.13. a-g

We prove 2.1.13.(e) $\forall A, B, C$ sets
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Next Page

Proof. <u>First we will show</u>

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

If $A \cap (B \cup C) = \emptyset$, then $\emptyset \subseteq (A \cap B) \cup (A \cap C)$. Thm 2.1.7

If $A \cap (B \cup C) \neq \emptyset$, then Let $x \in A \cap (B \cup C)$ be chosen arbitrarily.

$x \in A \cap (B \cup C)$

$x \in A$

$x \in B \cup C$

$x \in B$ OR $x \in C$

<u>Case 1</u> $x \in B$, <u>Case 2</u> $x \in C$

OR

$x \in B$	$x \in C$
$x \in A$ and $x \in B$	$x \in A$ and $x \in C$
$x \in A \cap B$.	$x \in A \cap C$

OR

$x \in A \cap B$ <u>OR</u> $x \in A \cap C$

$x \in (A \cap B) \cup (A \cap C)$

I showed that for any given $x \in A \cap (B \cup C)$,

I must have $x \in (A \cap B) \cup (A \cap C)$

(*) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

<u>Next</u>

I want to show

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

(Thm 2.1.7)

If $(A \cap B) \cup (A \cap C) = \emptyset$, then automatically we have

$$\emptyset \subseteq A \cap (B \cup C)$$

If $(A \cap B) \cup (A \cap C) \neq \emptyset$, then choose $x \in (A \cap B) \cup (A \cap C)$ arbitrarily.

$$x \in (A \cap B) \cup (A \cap C)$$

$x \in A \cap B$ OR $x \in A \cap C$

Case 1	Case 2
$x \in A \cap B$	$x \in A \cap C$
$x \in A$	$x \in A$
$x \in B$	$x \in C$

OR

In either case: $x \in B$ OR $x \in C$

$x \in (B \cup C)$

$x \in A$ (in both branches)

$x \in A$ and $x \in B \cup C$

$x \in A \cap (B \cup C)$

We established $\forall x \; \left(x \in (A \cap B) \cup (A \cap C) \implies x \in A \cap (B \cup C) \right)$

(**) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Combining (*) & (**) $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$. #

2.1.13 (f)

$\forall$ A, B, C sets : $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Proof $\forall x \in A$

$$
\begin{aligned}
x \in A \setminus (B \cup C) &\iff x \in A \text{ and } (x \notin B \cup C) \\
&\iff x \in A \text{ and not } (x \in B \cup C) \\
&\iff x \in A \text{ and not } (x \in B \text{ OR } x \in C) \\
&\iff x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\
&\iff (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\
&\iff x \in A \setminus B \text{ and } x \in A \setminus C \\
&\iff x \in (A \setminus B) \cap (A \setminus C).
\end{aligned}
$$

#

De Morgan's Laws

Take $A = \mathcal{U}$, $B, C \subseteq \mathcal{U} = A$.

$B^c = A \setminus B$

$C^c = A \setminus C$

Then (f) becomes

① $(B \cup C)^c = B^c \cap C^c$. ① is proved.

Next To Prove ② $(B \cap C)^c = B^c \cup C^c$:

With 2.1.13(c) $\mathcal{U} \setminus (\mathcal{U} \setminus B) = B$

which means $(B^c)^c = B$

one obtains ② from ① as follows :

Apply ① to B^c, C^c: $(B^c \cup C^c)^c = (B^c)^c \cap (C^c)^c = B \cap C$

$B^c \cup C^c = ((B^c \cup C^c)^c)^c = (B \cap C)^c$. $\therefore$ #