1.3

## Inductive reasoning

On the basis of looking at individual cases (many) to draw conclusions.

Useful in experimental sciences. but Not in Math

Ex (a) $\forall n \in \mathbb{N}$, $n^2+n+11$ is prime : TRUE OR FALSE?

| $n$ | $n^2+n+11$ |
|---|---|
| 1 | 13 |
| 2 | 17 |
| 3 | 23 |
| 4 | 31 |
| 5 | 41 |
| 6 | 53 |
| 7 | 67 |
| 8 | 83 |
| 9 | 101 |
| 10 | 121 |

prime for the first 9 tries.

121 not prime

statement is false

In Math we use deductive reasoning :

Applying a general principle to a particular case.

Exc 1.3.1

FALSE d) To prove $\forall n\ p(n)$ is true it takes one example

TRUE e) To prove $\exists n\ p(n)$ is true it takes one example

1.3.2

TRUE d) To prove $\forall n\ p(n)$ is false it takes one counterex.

FALSE e) To prove $\exists n\ p(n)$ is false it takes one counterex.

Ex i) $\forall x \in \mathbb{R}\quad x^2 > 0$ false since $\exists x = 0 \in \mathbb{R}$

$0^2 > 0$ is false

ii) $\exists x \in \mathbb{R}\quad x > 1$ true since $\exists x = 2 > 1$.

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Statement

$p \Rightarrow q$

$\sim q \Rightarrow \sim p$

contrapositive

equivalent to each other

BUT $\overset{?}{\longleftrightarrow}$

sometimes equivalent

sometimes not equivalent

$q \Rightarrow p$ converse

$\sim p \Rightarrow \sim q$

inverse

equivalent to each other

Ex (a) $x = 3 \Rightarrow x^2 = 9$ $\longleftrightarrow$ $x^2 \neq 9 \Rightarrow x \neq 3$

True True

$x^2 = 9 \Rightarrow x = 3$ $\longleftrightarrow$ $x \neq 3 \Rightarrow x^2 \neq 9$

false false c.ex $x = -3$

3.3 c

$f$ continuous and $C$ is connected $\Longrightarrow f(C)$ is connected

Contrapositive:

$f(C)$ is not connected $\Longrightarrow$ $f$ is not continuous OR $C$ is not connected.

3.6 Find counterexamples:

a) $\forall x \in \mathbb{R} \quad x^2 > 9 \Longrightarrow x > 3$

$x = -4$ is a counterexample

$(-4)^2 = 16 > 9$ Hypothesis ✓

$-4 > 3$ is false Conclusion is false.

(m) No rational number $x$ satisfies

$$x^3 + (x-1)^2 = x^2 + 1.$$

Work: $x^3 + \cancel{x^2} - 2x + \cancel{1} = \cancel{x^2} + \cancel{1}$

$x^3 - 2x = 0$

$x(x^2 - 2) = 0$

$x = 0$ is the only counterex.

$x = \pm\sqrt{2}$

3.7 abfh

Integers $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$

$m$ is even if $m = 2k$ for some $k \in \mathbb{Z}$

$m$ is odd if $m = 2k+1$ " " "

(a) $(p$ is odd & $q$ is odd $\Longrightarrow$ $p+q$ is even.$)$

Hidden quantifier $\forall p, q \in \mathbb{Z}$

Proof: $p$ is odd and $q$ is odd.

$p$ is odd

$p = 2k+1$ for some $k \in \mathbb{Z}$.

$q$ is odd.

$q = 2j+1$ for some $j \in \mathbb{Z}$.

$$\begin{aligned} p+q &= (2k+1) + (2j+1) \\ &= 2k + 2j + 2 \\ &= 2(k+j+1) \end{aligned}$$

$k, j \in \mathbb{Z} \Longrightarrow k+j+1 \in \mathbb{Z}$.

$p+q = 2 \cdot m$ where $m = k+j+1 \in \mathbb{Z}$.

$p+q$ is even.

QED.

(b) $\forall_{p,q \in \mathbb{Z}}$(p is odd & q is odd $\Longrightarrow$ pq is odd.)

Proof:

p is odd & q is odd.

p is odd

$p = 2k+1$ for some $k \in \mathbb{Z}$.

q is odd

$q = 2j+1$ for some $j \in \mathbb{Z}$.

$$p \cdot q = (2k+1)(2j+1)$$
$$= 4kj + 2k + 2j + 1$$
$$= 2(2kj + k + j) + 1$$

$m = 2kj + k + j$ is an integer

$$p \cdot q = 2m + 1$$

$p \cdot q$ is odd. QED

(h) If $p^2$ is even, then $p$ is even. ⊛

Direct proof involves Prime factorization Thm.

We will try to prove the contrapositive:

$p$ is not even $\Longrightarrow$ $p^2$ is not even

Need: ( $p$ is not even $\Longleftrightarrow$ $p$ is odd )

WTS: $p$ is odd $\Longrightarrow$ $p^2$ is odd.

We actually proved in (b) $p$ is odd + $q$ is odd $\longrightarrow$ $pq$ is odd.

By taking $p = q$

$p$ is odd $\Longrightarrow$ $p^2$ is odd. is a consequence of (b) which we proved.

Hence ⊛ is true, since its contrapositive is true.

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Given an integer $m$, We divide $m$ by 2. If $m$ is divisible by 2, then $m$ is even. If the remainder is 1, then $m$ is odd. There is no third possibility. Hence every integer is either odd or even, and not both.