1.2 Quantifiers

$\forall$ for all

$\exists$ there exists

} on a specific set

" $\forall x \in A,\ p(x)$ " means
"For all x in a set A, the proposition $p(x)$ holds (true)"

$\exists x \in A \ni p(x)$

$\exists x \in A \quad p(x)$

There exists an element x in the set A s.t. proposition $p(x)$ holds (true).

Exs

T $\exists x \in \mathbb{R} \ni x^2 = 5$

F $\forall x \in \mathbb{R} \quad x^2 = 5$

F $\exists x \in \mathbb{N} \ni x^2 = 5$

F $\exists x \in \mathbb{R} \ni x^2 = -1$

T $\exists x \in \mathbb{C} \ni x^2 = -1$

T $\forall x \in \phi \quad x = 1$ and x is a baloon.

F $\exists x \in \phi \quad x = 1$

$\phi = \{ \}$ empty set

* Orders of quantifiers are VERY important
* Always write quantifiers in the beginning

In Real #s:

True $\forall x \exists y \ni x < y$

False $\exists y \ni \forall x \quad x < y$

$\underbrace{\exists y \quad x < y \quad \forall x}_{\text{ambigious}}$

<u>Hidden quantifiers</u>

$$x \geq 4 \implies x^2 \geq 16$$

means

$$\underbrace{\text{“}\forall x\text{”}}_{\text{hidden}} \left(x \geq 4 \implies x^2 \geq 16 \right)$$

still not clear which set we are talking about

In a "If ... then" statement if the hypothesis contains variables, then it is assumed that $\forall$ applies (but which set A? It may be obvious from the context, & sometimes not.)

How to negate quantifiers:

$\sim [\forall x \in A, p(x)]$ is equivalent to

$\exists x \in A \ni \sim p(x)$

$\sim [\exists x \in A \ni q(x)]$ is equivalent to

$\forall x \in A, \sim q(x)$

Example 1 All books on this shelf are red.

negation:

There exists a book on this shelf, which is not red.

Example 2 $\sim \left(\exists x \in \mathbb{R} \ (x = 4) \right)$ is

$\forall x \in \mathbb{R} \quad x \neq 4$

Ex 3 $\forall x \in \mathbb{R} \ (x^2 - 9 = 0 \implies x = -3)$ False

negate

$\exists x \in \mathbb{R} \ \underbrace{(x^2 - 9 = 0 \text{ and } x \neq -3)}_{\text{true, I can take } x = 3}$

Ex. ④

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \ni \forall x \in \mathbb{R} \left(|x-3| < \delta \implies |x^2 - 9| < \varepsilon \right)$$

negate:

$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \in \mathbb{R} \left(|x-3| < \delta \text{ and } |x^2 - 9| \geq \varepsilon \right)$$

Exercises:

1.2 Exc #3 p16

d) $\exists x > 2 \ni f(x) = 7$

$\downarrow$ negate

$\forall x > 2 \quad f(x) \neq 7$

Assuming in $\mathbb{R}$.

e) $\forall x \in A, \exists y > 2 \ni \underbrace{0 < f(y) < f(x)}$

$0 < f(y)$ and $f(y) < f(x)$

(Assume $A = \mathbb{R} = f(A)$)

negate

$$\exists x \in A \ \forall y > 2 \ \left(0 \geq f(y) \text{ OR } f(y) \geq f(x) \right)$$

Caution: in $\mathbb{R}$: not$(x<y)$ $\iff$ $x \geq y$

In some other partially ordered sets, this is NOT true

f) If $x > 3$, then $\exists \varepsilon > 0 \ni x^2 > 9 + \varepsilon$

negate

(Assume in $\mathbb{R}$)

$x > 3$ and not $(\exists \varepsilon > 0 \ni x^2 > 9 + \varepsilon)$

$\iff$ $x > 3$ and $\forall \varepsilon > 0 \quad x^2 \leq 9 + \varepsilon$.

p17 1.2 Exc #5 All in $\mathbb{R}$:

T (a) $\exists x \in [2,4] \quad x<7$

T (b) $\forall x \in [2,4] \quad x<7$

T (c) $\exists x \ni \quad x^2=5$

F (d) $\forall x \quad x^2=5$ $\quad$ ($x=0 \in \mathbb{R}$ but $0^2 \neq 5$)

T (e) $\exists x \quad x^2 \neq -3$

T (f) $\forall x \quad x^2 \neq -3$ $\quad$ (False in $\mathbb{C}$)(True in $\mathbb{R}$)

T (g) $\exists x \quad x \div x = 1$ $\quad$ ($x=1$ is an example)

F (h) $\forall x \quad x \div x = 1$ $\quad$ $0 \div 0$ is undefined $\neq 1$.

p17 1.2 Exc #9 all x, y real $\mathbb{R}$

(a) $\underbrace{\forall x \text{ and } y}$

F $\forall x \forall y \quad x \leq y$ $\quad$ (since $\overset{x}{\overset{\shortparallel}{2}}, \overset{y}{\overset{\shortparallel}{1}} \in \mathbb{R}$ & $2 \not\leq 1$)

(b) $\underbrace{\exists \ x \ \& \ y}$

T $\exists x \exists y \quad x \leq y$ $\quad$ ($\exists x=5 \ \exists y=6$ & $5 \leq 6$)

(c) $\forall x \exists y \quad x \leq y$

T True since for any given x, take $y=x+1$.
or take $y=x$.

F (d) $\exists x \forall y \quad x \leq y$

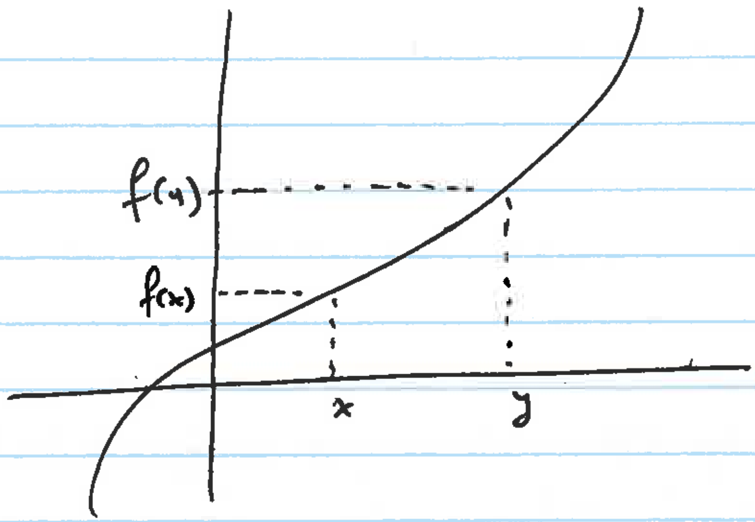
There is no smallest number in $\mathbb{R}$

However True in $\mathbb{N}$. $\exists x=1 \in \mathbb{N} \ \forall y \in \mathbb{N}, \ 1 \leq y$.

Exc #15

Increasing function: $f: \mathbb{R} \longrightarrow \mathbb{R}$

$\forall x \, \forall y \in \mathbb{R}$ if $x \leq y$ then $f(x) \leq f(y)$



Negate

$\exists x \, \exists y \, (\, x \leq y$ and $f(x) > f(y) \,)$