May 7, 2020

7.2 Continue

①

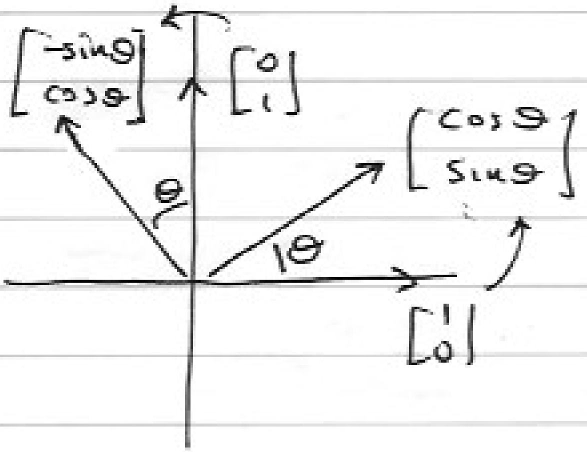
__Examples__ orthogonal matrices in $\mathbb{R}^2$

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{P \text{ orthogonal}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Rotation counterclockwise for θ radians, If we consider $P.v = w$ as a function acting on fixed coordinates

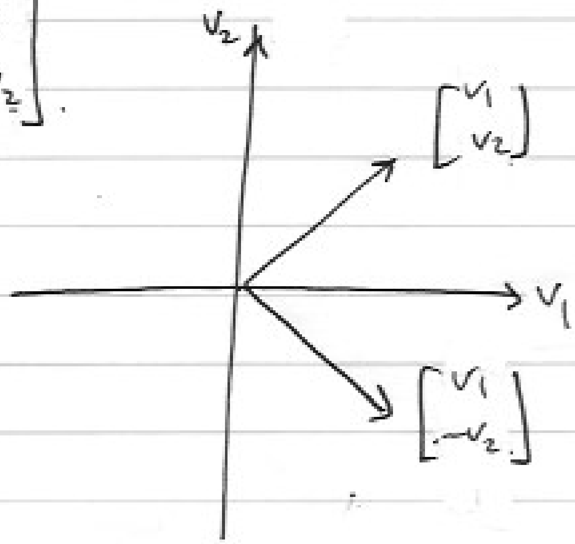
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$



$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{orthogonal matrix.}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}.$$

reflection across v_1 axis



Example

$$Q(x_1, x_2) = 9x_1^2 - 8x_1x_2 + 3x_2^2$$

1. Make change of variables to obtain a quadratic form with no cross terms

$$A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$$

Orthogonally diagonalize A

$$\begin{vmatrix} 9-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} = (9-\lambda)(3-\lambda) - 16$$

$$= 27 - 12\lambda + \lambda^2 - 16$$

$$= \lambda^2 - 12\lambda + 11$$

$$\begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$$

$a_{11} + a_{22} = 9 + 3 = \text{trace} = \lambda_1 + \lambda_2$

$\det = \lambda_1 \lambda_2$

$$\begin{vmatrix} 9 & -4 \\ -4 & 3 \end{vmatrix} = 11$$

$$\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$$

$\lambda_1 = 1$

$\lambda_2 = 11$

$\lambda = 1$ eigenspace

$$A - I = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$x_1 - \frac{1}{2}x_2 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

basis or use $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
a multiple

$\lambda = 11$

$$A - 11I = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ orthogonal basis

length $\sqrt{5}$ $\sqrt{5}$

$\left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$ orthonormal basis

$$\underbrace{\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}}_{P} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

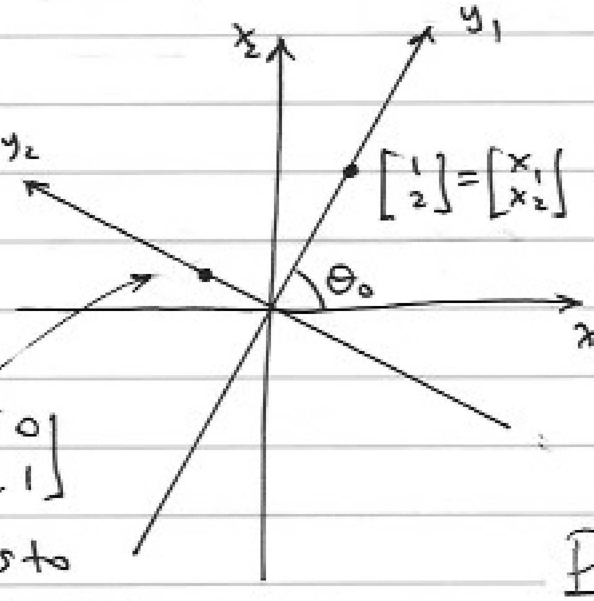
$$y = x$$

Observe that $\exists\, \theta_0$ angle s.t. $\cos\theta_0 = \frac{1}{\sqrt{5}}$

$\sin\theta_0 = \frac{2}{\sqrt{5}}$

P is a rotation matrix. (no reflection in this example)

$$Q = 9x_1^2 - 8x_1x_2 + 3x_2^2 = y_1^2 + 11y_2^2$$



$\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ corresponds to $\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}\sqrt{5}\\0\end{bmatrix}$.

via $x = Py$

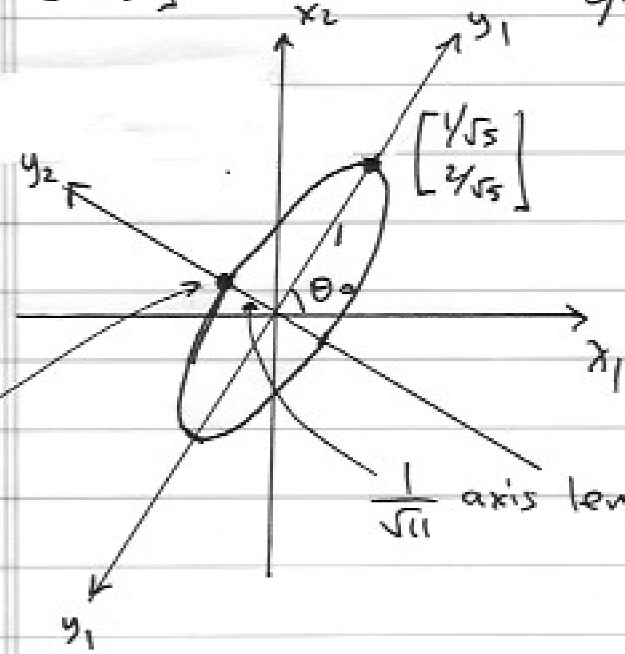
$$P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ corresponds to $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$

<u>Part 2</u>

graph

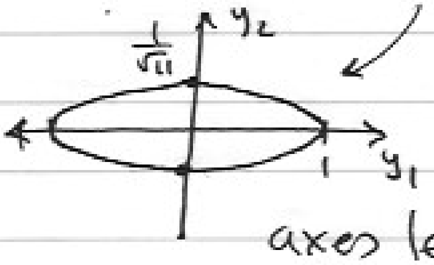
$$9x_1^2 - 8x_1x_2 + 3x_2^2 = y_1^2 + 11y_2^2 = 1$$



$\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$

$\frac{1}{\sqrt{11}}\underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\end{bmatrix}}_{\text{unit}}$

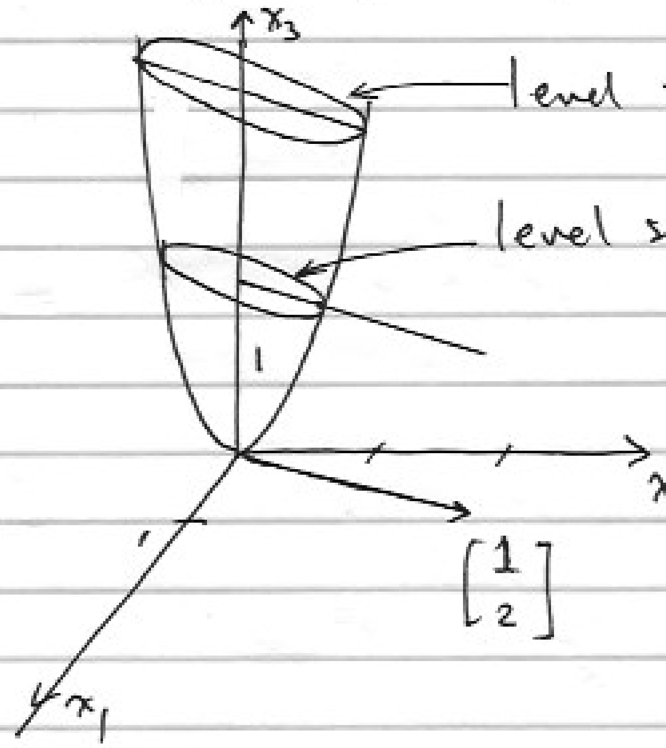
$\frac{1}{\sqrt{11}}$ axis length.



axes lengths $1, \frac{1}{\sqrt{11}}$

Part 3 Plot 3-D graph

$$x_3 = 9x_1^2 - 8x_1x_2 + 3x_2^2$$



level set for $x_3 = 2$

level set for $x_3 = 1 = 9x_1^2 - 8x_1x_2 + 3x_2^2$

Part 4 What is the largest value of $9x_1^2 - 8x_1x_2 + 3x_2^2$ with the constraint $x_1^2 + x_2^2 = 1$

Same as as $y_1^2 + 11y_2^2$ with $y_1^2 + y_2^2 = 1$

Answer: 11

Why are they the same?

under $Py = x$ one circle goes to the other.

CLASSIFYING QUADRATIC FORMS

A quadratic form is called positive definite if $Q(x) > 0$ for all $x \neq 0$

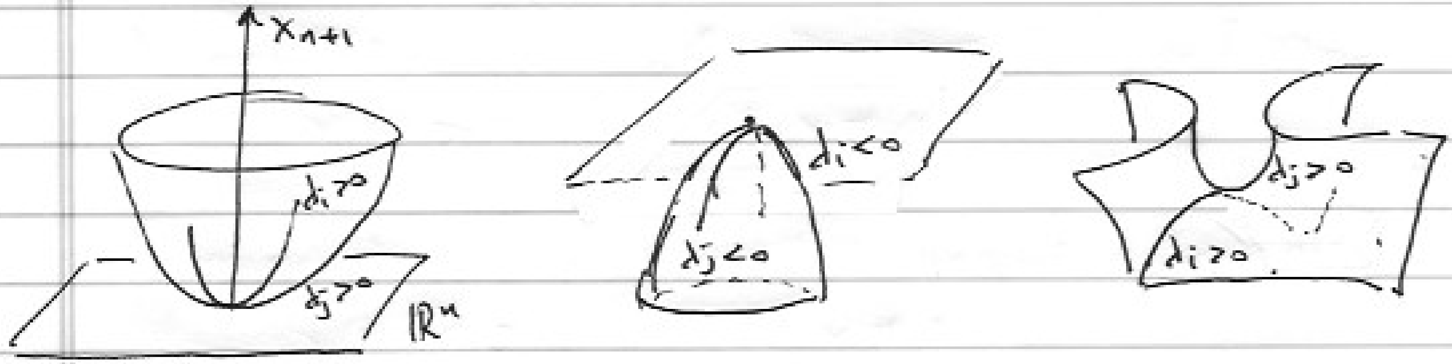
A quadratic form is called negative definite if $Q(x) < 0$ for all $x \neq 0$

A quadratic form is called indefinite if
$$Q(x) < 0 < Q(y)$$
for some x, y both non-zero.

<u>Theorem</u>: Let A be a symmetric matrix. $Q(x) = x^T A x$ is

- positive definite $\iff$ all eigenvalues of A are positive
- negative definite $\iff$ all eigenvalues of A are negative
- indefinite $\iff$ A has both positive and negative eigenvalues.

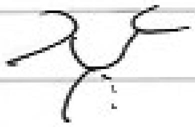
Graphs $x_{n+1} = x^T A x$ (A is $n \times n$.)



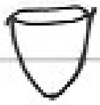
positive definite negative definite indefinite

$\underline{\underline{Ex}}$ $\begin{bmatrix} a & b \\ b & d \end{bmatrix} = A$ When is $Q(x) = x^T A x$ $+, -,$ indefinite

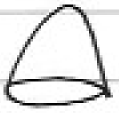
$$\lambda_1 \lambda_2 = \det = ad - b^2$$
$$\lambda_1 + \lambda_2 = a + d$$



$Q(x)$ is indefinite $\iff$ $ad - b^2 < 0$



$Q(x)$ is positive definite $\iff$ $a + d > 0$



$Q(x)$ is negative definite $\iff$ $a + d < 0$

$\Big\}$ both $ad - b^2 > 0$