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# Continue 7.1

**Example** ~ 24

$$A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}, \text{ given eigenvectors } \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Orthogonally diagonalize

**Solution**

$$\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \qquad \lambda_1 = 10$$

$$\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_2 = 1.$$

What is $\lambda_3 = ?$

$$\begin{vmatrix} 5-\lambda & -4 & -2 \\ -4 & 5-\lambda & 2-\lambda \\ -2 & 2 & 2-\lambda \end{vmatrix} = \text{Long to calculate \& I know 2 eigenvalues.}$$

$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$$

$$= -\lambda^3 + \underbrace{(\lambda_1 + \lambda_2 + \lambda_3)}\lambda^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda + \underbrace{\lambda_1\lambda_2\lambda_3}$$

$a_{11} + a_{22} + a_{33} = 12 = \text{trace } A \longrightarrow \lambda_3 = 1 \longleftarrow \quad 10 = \det A$

$$\begin{vmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 9 & 2 \\ 0 & 4 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 9 & 2 \\ 4 & 2 \end{vmatrix} = 10 = \det A = \lambda_1 \lambda_2 \lambda_3$$

E. values $10, 1, 1$ $\quad \lambda = 10$: multiplicity 1

$\lambda = 1$: multiplicity 2

eigenspace for $\lambda = 10$ No need to RR $A - 10I$, since multiplicity of 10 is 1

$\Rightarrow$ dimension eigenspace for $\lambda = 10$ is 1

We know $\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ is in this eigenspace, use it

$\underbrace{\text{Eigenspace for } \lambda = 1:}_{}$ We have $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, need another basis vector

dimension is 2 since multiplicity is 2

$$A - I = \begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \quad x_1 - x_2 - \tfrac{1}{2} x_3 = 0$$

pivot free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + \frac{1}{2} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} +\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{cccccc} \lambda_1 = 10 & & \lambda_2 = 1 & & \lambda_3 = 1 & \\ \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right. & , & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & , & \left. \begin{bmatrix} +\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \\ u_1 & & u_2 & & u_3 & \end{array}$$

A basis with eigenvectors

$\left. \begin{array}{l} u_1 \cdot u_2 = 0 \\ u_1 \cdot u_3 = 0 \end{array} \right\}$ from different eigenspaces.

Apply Gram-Scmidt to $u_2$ $u_3$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} ; \quad v_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} - \frac{1/2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -1/4 \\ 1 \end{bmatrix}$$

can use any multiple (non-zero)

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}$$ orthogonal basis

lengths $3$ $\quad \sqrt{2}$ $\quad \sqrt{18} = 3\sqrt{2}$

need orthonormal

$$P = \begin{bmatrix} -2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \\ 2/3 & 1/\sqrt{2} & -1/3\sqrt{2} \\ 1/3 & 0 & 4/3\sqrt{2} \end{bmatrix} \qquad D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

You may want to rationalize the denominators: $1/3\sqrt{2} = \frac{\sqrt{2}}{6}$ etc.

Spectral Decomposition:

$$A = PDP^T =$$

$$A = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}}_{\text{column vectors}} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}}_{\text{row-vectors}}$$

projection matrices

$$= \underbrace{\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T}_{\text{Called spectral decomposition}}$$

<u>Example</u> We did $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ : $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $D = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}$

$$u_1 u_1^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$u_2 u_2^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$-4 u_1 u_1^T + 6 u_2 u_2^T = -4 \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 6 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

# 7.2 QUADRATIC FORMS

**Defn** Let $A$ be an $n \times n$ symmetric matrix

Define $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$Q(\vec{x}) = x^T A x$$

$Q$ is called the quadratic form associated to $A$.

**Ex 0**

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{x_1^2 + x_2^2 + x_3^2}_{\vec{x} \cdot \vec{x}}$$

**Ex 1**

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$x^T A x = 2x_1^2 + 5x_2^2 - 3x_3^2$$

**Ex 2**

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 5 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_2 + 3x_3 & x_1 - x_2 + 5x_3 & 3x_1 + 5x_2 + 4x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= (2x_1^2 + x_1x_2 + 3x_1x_3) + (x_2x_1 - x_2^2 + 5x_2x_3) + \\ + (3x_3x_1 + 5x_3x_2 + 4x_3^2)$$

$$= 2x_1^2 - x_2^2 + 4x_3^2 + \underline{\underline{2}}(x_1x_2 + 3x_1x_3 + 5x_2x_3)$$

Compare:
$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 5 \\ 3 & 5 & 4 \end{bmatrix}$$

Easily doable backwards

In $\mathbb{R}^2$ $\quad 3x_1^2 + 5x_2^2 - 7x_1x_2 \longrightarrow \begin{bmatrix} 3 & -\frac{7}{2} \\ -\frac{7}{2} & 5 \end{bmatrix}$

Caution

In $\mathbb{R}^3$ $\quad 3x_1^2 + 5x_2^2 - 7x_1x_2 \longrightarrow \begin{bmatrix} 3 & -\frac{7}{2} & 0 \\ -\frac{7}{2} & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In $\mathbb{R}^3$ $\quad 3x_1^2 + 6x_1x_2 - 7x_2^2 + 3x_1x_3 + 8x_2x_3$ :

$$\begin{bmatrix} 3 & 3 & 3/2 \\ 3 & -7 & 4 \\ 3/2 & 4 & 0 \end{bmatrix}$$

CHANGE of VARIABLES in Quadratic Forms.

$x = Py$

$x^T A x = (Py)^T A Py = y^T P^T A P y$

A symmetric if we find P s.t.

$A = PDP^T$

for D diagonal

P orthogonal ($P^T = P^{-1}$)

then $P^T A P = P^T (PDP^T) P$

$= (P^T P) \cdot D (P^T P) = D$.

diagonal.

PRINCIPAL AXIS THEOREM

Let A be a $n \times n$ symmetric matrix. Then there exists an orthogonal change of variable $x = Py$ that transforms the quadratic form $x^T A x$ into a quadratic form $y^T D y$ which has no cross-product terms.

Example

$$Q(x) = x_1^2 + 10x_1x_2 + x_2^2$$

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad D = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}$$

$$x = Py \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}}(-y_1 + y_2)$$

$$x_2 = \frac{1}{\sqrt{2}}(y_1 + y_2)$$

$$x_1^2 + 10x_1x_2 + x_2^2 =$$

$$= \frac{1}{2}(y_2 - y_1)^2 + 10 \cdot \frac{1}{2}(y_2 - y_1)(y_2 + y_1) + \frac{1}{2}(y_1 + y_2)^2$$

$$= \frac{1}{2}(y_2^2 - \cancel{2y_1y_2} + y_1^2) + \frac{1}{2}(10y_2^2 - 10y_1^2) + \frac{1}{2}(y_1^2 + \cancel{2y_1y_2} + y_1^2)$$

$$= y_2^2\left(\frac{1}{2} + 5 + \frac{1}{2}\right) + y_1^2\left(\frac{1}{2} - 5 + \frac{1}{2}\right)$$

$$= -4y_1^2 + 6y_2^2 \qquad \text{compare to } D = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}.$$