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CHAP 7

7.1 DIAGONALIZATION of SYMMETRIC MATRICES

Recall A^T: writing the rows of A for columns of A^T.

Def^n A is called symmetric if $A^T = A$

Ex $\begin{bmatrix} 1 & 3 & 4 \\ 3 & -2 & 5 \\ 4 & 5 & 0 \end{bmatrix}$ symmetric

$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is not symmetric; $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

Defn Orthogonal matrix: $PP^T = P^TP = I$
that is $P^T = P^{-1}$.

Recall Thm 6 of 6.2

Prop $PP^T = P^TP = I$

$\iff$ columns of P form an orthonormal basis for $\mathbb{R}^n$

$\iff$ Rows of P form an orthonormal basis for $\mathbb{R}^n$.

Ex

$$\begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \end{matrix} \\ \begin{matrix} u_1 \rightarrow \\ u_2 \rightarrow \\ u_3 \rightarrow \end{matrix} \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} & \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} = P^T P \end{array}$$

$$(*) \quad = \begin{bmatrix} \frac{4}{6}+\frac{1}{6}+\frac{1}{6} & -\frac{2}{\sqrt{18}}+\frac{1}{\sqrt{18}}+\frac{1}{\sqrt{18}} & 0-\frac{1}{\sqrt{12}}+\frac{1}{\sqrt{12}} \\ -\frac{2}{\sqrt{18}}+\frac{1}{\sqrt{18}}+\frac{1}{\sqrt{18}} & \frac{1}{3}+\frac{1}{3}+\frac{1}{3} & 0-\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}} \\ 0-\frac{1}{\sqrt{12}}+\frac{1}{\sqrt{12}} & 0-\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}} & 0+\frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

You can also check $u_1 \cdot u_2 = u_1 \cdot u_3 = u_2 \cdot u_3 = 0$
$u_1 \cdot u_1 = u_2 \cdot u_2 = u_3 \cdot u_3 = 1$
but that is already done in $(*)$

<u>Thm: 1</u> : For A is symmetric, two eigenvectors from different eigenspaces are orthogonal.

$$\left.\begin{matrix} Av_1 = \lambda v_1, & A^T = A \\ Av_2 = \mu v_1, & \mu \neq \lambda \end{matrix}\right\} \Longrightarrow v_1 \cdot v_2 = 0$$

Matrix multiplication

$$\lambda (v_1 \cdot v_2) = (\lambda v_1) \cdot v_2 = (\lambda v_1)^T v_2 = (Av_1)^T v_2 = (v_1^T A^T)\, v_2 = (v_1^T)(A^T v_2)$$

dot product

$$= (v_1^T)(Av_2) = v_1^T \mu v_2 = \mu\,(v_1^T v_2) = \mu\, v_1 \cdot v_2$$

$$\lambda \neq \mu \ \& \ \lambda (v_1 \cdot v_2) = \mu (v_1 \cdot v_2) \Longrightarrow \quad v_1 \cdot v_2 = 0$$

*** CRUCIAL THM 2 Let A be an $n \times n$ matrix.

$$\underbrace{A \text{ is symmetric}}_{A^T = A} \iff \underbrace{A \text{ is orthogonally diagonalizable}}_{\text{Def}^n \; A = PDP^T \text{ where}}$$

- D is diagonal
- P is orthogonal $PP^T = P^TP = I$.

*** SPECTRAL THM Let A be an $n \times n$ symmetric matrix. Then

(i) A has n real eigenvalues if counted with multiplicities.

(ii) For each eigenvalue λ of A, dim of eigenspace for λ = multiplicity of λ

(iii) Eigenspaces of different eigenvalues are orthogonal

(iv) A is orthogonally diagonalizable.

Converse also true

Orthogonally diagonalizable $\Rightarrow$ symmetric

$A = PDP^T$

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = P \underset{\uparrow \text{diagonal}}{D^T} P^T = \underset{\underset{A.}{\parallel}}{PDP^T}$$

Examples $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ symmetric.

Orthogonally diagonalize:
Find P, D s.t.
$PDP^T = A$ · P orthogonal
· D diagonal.

$$\begin{vmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 25 = 1 - 2\lambda + \lambda^2 - 25$$
$$= \lambda^2 - 2\lambda - 24$$
$$= (\lambda - 6)(\lambda + 4)$$

$\lambda = -4, 6$ e.values

$\lambda = -4$ eigenspace

$$A - \lambda I = A + 4I = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

pivot free

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

basis for eigenspace -4

$\lambda = 6$ eigenspace

$$A - 6I = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

pivot free

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

basis for eigenspace $\lambda = 6$

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$ orthogonal basis for $\mathbb{R}^2$

$$\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$ orthonormal basis for $\mathbb{R}^2$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad D = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}$$

Check $A = PDP^T$

$$\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{+1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{\sqrt{2}} & \frac{6}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} & \frac{6}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{2} + \frac{6}{2} & \frac{4}{2} + \frac{6}{2} \\ \frac{4}{2} + \frac{6}{2} & -\frac{4}{2} + \frac{6}{2} \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}.$$

<u>Ex</u> Orthogonally diagonalize $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \begin{matrix} 1-\lambda & 1 \\ 1 & 1-\lambda \\ 1 & 1 \end{matrix}$$

$$= \left((1-\lambda)^3 + 1 + 1\right) - (1-\lambda) - (1-\lambda) - (1-\lambda)$$

$$= 1 - 3\lambda + 3\lambda^2 - \lambda^3 + 2 - 3 + 3\lambda$$

$$= -\lambda^3 + 3\lambda^2 = \lambda^2(-\lambda+3)$$

Eigenvalues 0, 0, 3

$\lambda = 0$ $\quad A - 0I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{array}{cc} & \uparrow \text{free} \ \uparrow \text{free} \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{basis for } \lambda = 0 \text{ e-value}}.$$

$$\left.\begin{matrix}\text{dim eigenspace} \\ \text{for } \lambda = 0\end{matrix}\right\} = 2 = \text{multiplicity of } 0$$

$\lambda = 3 \quad A - 3I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

$$\xrightarrow[R_3 + 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow[\frac{1}{3}R_3]{R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot

free x_3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\left\{ \underset{u_1}{\overset{\lambda=0}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}}, \underset{u_2}{\overset{\lambda=0}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}}, \underset{u_3}{\overset{\lambda=3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}} \right\}$$ a basis of eigenvectors for $\mathbb{R}^3$

Is it orthogonal? NO

$\left. \begin{array}{l} u_1 \cdot u_3 = 0 \\ u_2 \cdot u_3 = 0 \end{array} \right\}$ coming from distinct eigenvalues.

$u_1 \cdot u_2 = 1 \neq 0$

Apply Gram-Schmidt to u_1, u_2 in $W =$ eigenspace for $\lambda = 0$

$V_1 = u_1$

$$V_2 = u_2 - \frac{u_2 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \overbrace{\frac{1+0+0}{1+1+0}}^{1/2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+\frac{1}{2} \\ 0-\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\left\{ \overset{v_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}}, \overset{u_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$$ orthogonal basis for $\mathbb{R}^3$

length $\downarrow$ $\sqrt{2}$ length $\downarrow$ $\sqrt{\frac{1}{4}+\frac{1}{4}+1} = \sqrt{\frac{3}{2}}$ $\rightarrow \sqrt{3}$

divide each by its length

$$\left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}$$ orthonormal. basis for $\mathbb{R}^3$

orthogonal matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$PDP^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = A.$$

Summarize Procedure:

- Given $n \times n$ symmetric matrix A.
- Calculate $\det(A - \lambda I)$
- Find all eigenvalues. All real, exactly n counting with multiplicities.
- For each eigenvalue, find a basis for the corresponding eigenspace.
- Use Gram-Schmidt process to orthogonalize the basis for each eigenspace seperately. (Recall Thm 1 p 2)
- Combine all, to find an orthogonal basis for $\mathbb{R}^n$
- Normalize the basis to obtain an orthonormal basis: $v_1, v_2 \cdots v_n$.

orthogonal matrix $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$ diagonal

column vectors

$$Av_i = \lambda_i v_i \quad i = 1, 2, \cdots n$$

$$A = PDP^T.$$