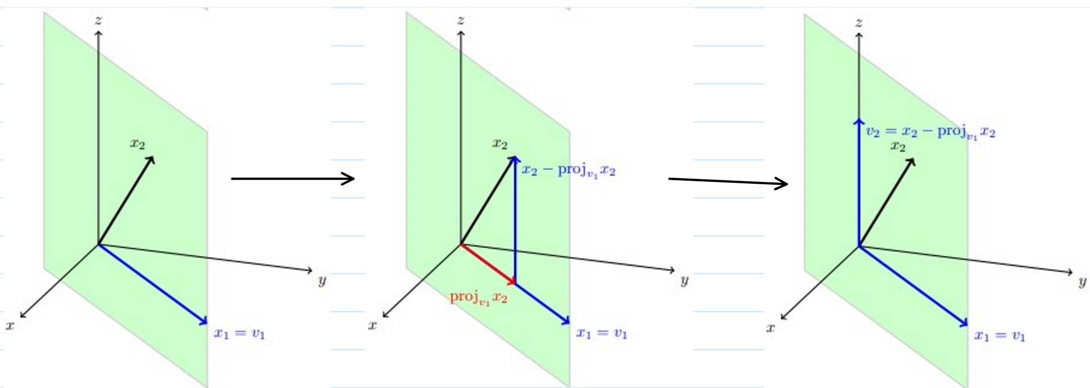
April 28, 2020

Section 6.4

Example 1 Let $W = \text{span}\{x_1, x_2\}$ where $x_1 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$. Find an orthogonal basis $\{v_1, v_2\}$ for W.

Solution:



$$v_1 = x_1 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

$$v_2 = x_2 - \text{proj}_{v_1} x_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}}{\begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}} \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{8+8}{16+16}\right) \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \frac{16}{32} \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

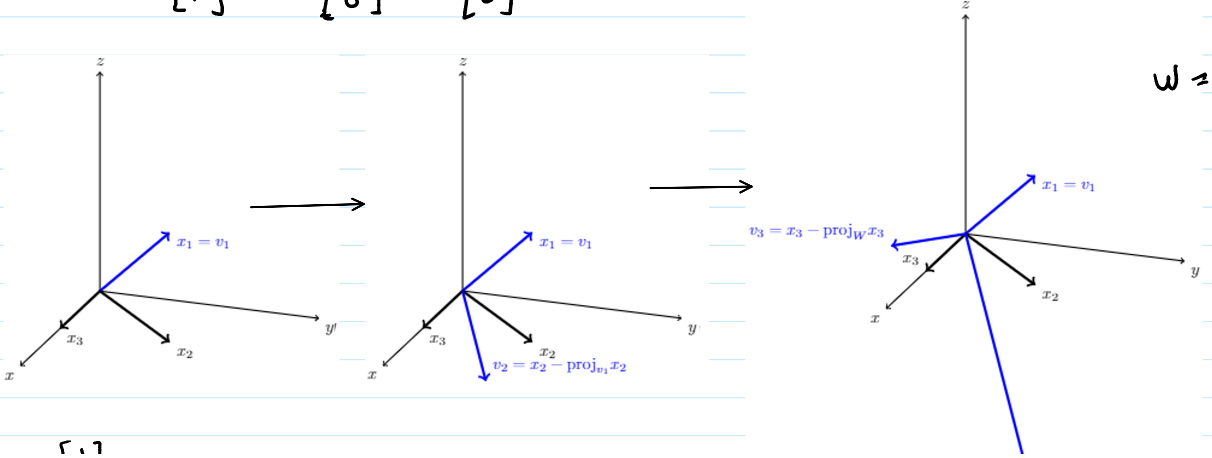
$$= \begin{bmatrix} 2-2 \\ 2-2 \\ 3-0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

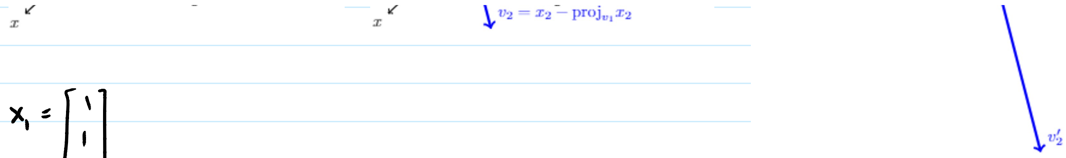
Note that $v_2 \in \text{span}\{x_2, v_1\} = \text{span}\{x_1, x_2\} = W$ so $\{v_1, v_2\} \subseteq W$ is a orthogonal set in W. Since $\dim W = 2$ this gives that $\{v_1, v_2\} = \left\{ \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ is an orthogonal basis for W.

Example 2 Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. These vectors span $\mathbb{R}^3$. Find an orthogonal basis containing x_1 for $\mathbb{R}^3$.

Solution:



$W = \text{span}\{v_1, v_2\}$



① $v_1 = x_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

② $v_2 = x_2 - \mathrm{proj}_{v_1} x_2 = x_2 - \dfrac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \dfrac{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \left(\dfrac{1+1}{1+1+1}\right)\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \dfrac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$= \begin{bmatrix}1-2/3\\1-2/3\\0-2/3\end{bmatrix}$

$= \begin{bmatrix}1/3\\1/3\\-2/3\end{bmatrix}$

(optional) $v_2' = 3v_2 = \begin{bmatrix}1\\1\\-2\end{bmatrix}$

③ $v_3 = x_3 - \mathrm{proj}_W x_3 = x_3 - \left(\dfrac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 + \dfrac{x_3 \cdot v_2'}{v_2' \cdot v_2'} v_2'\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} - \dfrac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} - \dfrac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}{\begin{bmatrix}1\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$

$= \begin{bmatrix}1\\0\\0\end{bmatrix} - \dfrac{1}{1+1+1}\begin{bmatrix}1\\1\\1\end{bmatrix} - \dfrac{1}{1+1+4}\begin{bmatrix}1\\1\\-2\end{bmatrix}$

$= \begin{bmatrix}1\\0\\0\end{bmatrix} - \dfrac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \dfrac{1}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix}$

$= \begin{bmatrix}1-1/3-1/6\\0-1/3-1/6\\0-1/3+1/3\end{bmatrix}$

$= \begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}$

$\{v_1, v_2', v_3\} = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}\right\}$

Theorem 11 (The Gram-Schmidt Process)

Given a basis $\{x_1, \dots, x_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$\vdots$$

$$v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$$

then $\{v_1, \dots, v_p\}$ is an orthogonal basis for W. In addition, $\text{span}\{v_1, \dots, v_k\} = \text{span}\{x_1, \dots, x_k\}$ for all $1 \le k \le p$.

Example 3 Let $W = \text{span}\{x_1, x_2, x_3\}$ where $x_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix}$. Find an orthogonal basis for W.

Solution: ① $v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

② $$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - \left(\frac{1+4+3}{1+1+9}\right) \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - 8/11 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 8/11 \\ 4 - 8/11 \\ 1 - 24/11 \end{bmatrix}$$

$$= \begin{bmatrix} 3/11 \\ 36/11 \\ -13/11 \end{bmatrix}$$

(optional) $v_2' = 11 v_2 = \begin{bmatrix} 3 \\ 36 \\ -13 \end{bmatrix}$

③ $$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2'}{v_2' \cdot v_2'} v_2' = \begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 36 \\ -13 \end{bmatrix}}{\begin{bmatrix} 3 \\ 36 \\ -13 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 36 \\ -13 \end{bmatrix}} \begin{bmatrix} 3 \\ 36 \\ -13 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix} - \left(\frac{1+10-9}{1+1+9}\right) \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \left(\frac{3+360+39}{9+1296+169}\right) \begin{bmatrix} 3 \\ 36 \\ -13 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix} - \frac{2}{11} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \overbrace{\frac{402}{1474}}^{3/11} \begin{bmatrix} 3 \\ 36 \\ -13 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2/11 - 9/11 \\ 10 - 2/11 - \frac{108}{11} \\ -3 - 6/11 + \frac{39}{11} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{11} - 2/11 - 9/11 \\ \frac{110}{11} - 2/11 - \frac{108}{11} \\ \frac{-33}{11} - 6/11 + \frac{39}{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_3 \in \text{Span}\{v_1, v_2'\} = \text{Span}\{x_1, x_2\}$

An orthogonal basis for W is $\{v_1, v_2'\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 34 \\ -13 \end{bmatrix} \right\}$.