6.3

Recall **Defn** Let W be a subspace of $\mathbb{R}^n$.

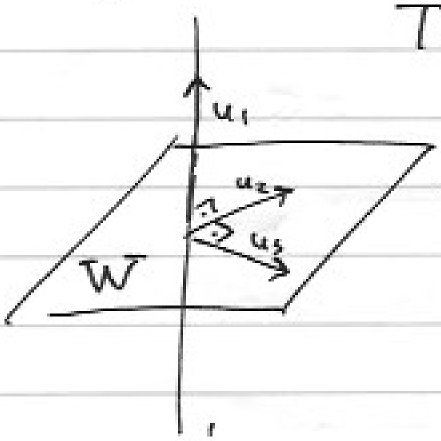
$W^{\perp}$ is the collection of all vectors $\vec{z}$ in $\mathbb{R}^n$ such that $\vec{z}\cdot\vec{w}=0$ for all $w \in W$.

$W^{\perp}$ is called the orthogonal complement of W.

Ex If $\{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$

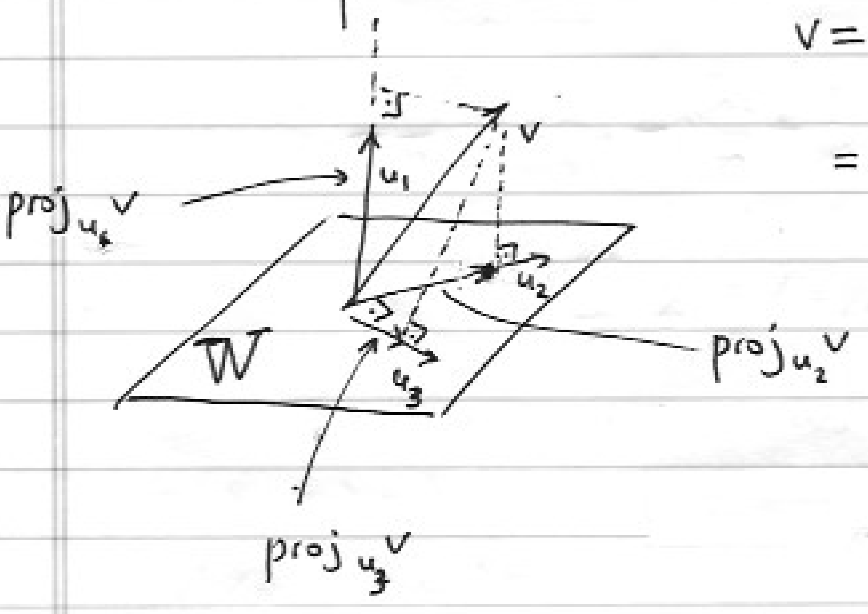
Then $W = \text{span}\{\vec{u}_2, \vec{u}_3\} = V^{\perp}$

$W^{\perp} = \text{span}\{u_1\}. = V$



$$v = \text{proj}_{u_1} v + \text{proj}_{u_2} v + \text{proj}_{u_3} v$$

$$= \underbrace{\frac{u_1\cdot v}{u_1\cdot u_1}u_1}_{\text{in } W^{\perp}} + \underbrace{\frac{u_2\cdot v}{u_2\cdot u_2}u_2 + \frac{u_3\cdot v}{u_3\cdot u_3}u_3}_{\text{in } W}$$



Example

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} \quad u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

Given $\{u_1, u_2, u_3, u_4\}$ orthogonal set.

$x = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$, write $x = v_1 + v_2$ s.t.

v_1 in Span $\{u_1, u_2, u_3\}$
v_2 in Span $\{u_4\}$

$$x \cdot u_1 = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = 14 \qquad x \cdot u_2 = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = 3$$

$u_1 \cdot u_1 = 7$ $\qquad$ $u_2 \cdot u_2 = 7$

$$x \cdot u_3 = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} = 12 \qquad x \cdot u_4 = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} = -8$$

$u_3 \cdot u_3 = 7$ $\qquad$ $u_4 \cdot u_4 = 7$

$$\vec{X} = \underbrace{\frac{14}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{7} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{12}{7} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}} + \underbrace{\frac{-8}{7} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}}$$

$$\begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} = \vec{X} = \underbrace{\begin{bmatrix} 20/7 \\ 43/7 \\ -13/7 \\ 5/7 \end{bmatrix}}_{v_1} + \underbrace{\begin{bmatrix} 8/7 \\ -8/7 \\ -8/7 \\ 16/7 \end{bmatrix}}_{v_2}$$

Theorem 8

Let W be a subspace of $\mathbb{R}^n$.
Each y in $\mathbb{R}^n$ can be written uniquely in the form

$$y = \hat{y} + z \quad \text{where}$$

$$\hat{y} \text{ in } W, \text{ and } z \text{ in } W^{\perp}$$

If $\{u_1, u_2, \ldots u_p\}$ is an orthogonal basis for W then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p.$$

and $\quad z = y - \hat{y}$

NOTATION:

$$\hat{y} = \text{proj}_W y = \sum_{i=1}^{p} \frac{y \cdot u_i}{u_i \cdot u_i} u_i$$

Exc #5

$$y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

Check u_1, u_2 orthogonal.

Find $\text{proj}_W y$ where $W = \text{span}\{u_1, u_2\}$.

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = 3 + 1 - 4 = 0 \qquad u_1, u_2 \text{ orthogonal.}$$

$$y \cdot u_1 = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = -3 - 2 + 12 = 7 \qquad u_1 \cdot u_1 = 9 + 1 + 4 = 14$$

$$y \cdot u_2 = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = -1 - 2 - 12 = -15 \qquad u_2 \cdot u_2 = 6$$

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{7}{14}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \frac{15}{6}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} = \text{proj}_W y = y$$

$$z = y - \hat{y} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} = 0 \qquad \text{since } y \in W.$$

Ex #8 $y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$ $\underbrace{u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}}$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = -1+3-2 = 0$$ orthogonal ✓

$$y \cdot u_1 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1+4+3 = 6 \qquad u_1 \cdot u_1 = 3$$

$$y \cdot u_2 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = 1+12-6 = 7 \qquad u_2 \cdot u_2 = 14$$

$$\text{proj}_W\, y = \frac{6}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{14}\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} = \hat{y}$$

$W = \text{span}\{u_1, u_2\}$

$$z = y - \hat{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$$

z is in $W^{\perp}$

Check $z \cdot u_1 = 0$? $\qquad z \cdot u_2 = 0$?

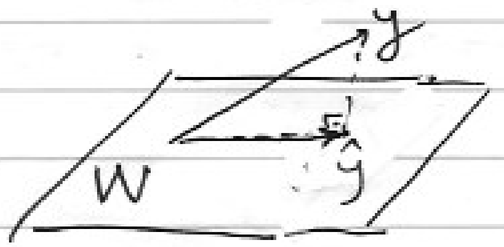
$$\begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \checkmark \qquad \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = 0 \checkmark$$

THEOREM 9 Best Approximation Thm

Let W be a subspace of $\mathbb{R}^n$, y be in $\mathbb{R}^n$.

Let $\hat{y} = \text{proj}_W y$. Then $\hat{y}$ is the closest point of W to y:

$$\|y-\hat{y}\| < \|y-v\| \text{ for all } v \text{ in } W, \; v \neq y$$



Exc #15 Find the closest pt of the plane W spanned by u_1 and u_2 to y.
Find distance from y to W.

$$y = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}, \quad \underbrace{u_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}, \; u_2 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}}_{\text{check orthogonal}}.$$

$u_1 \cdot u_2 = 9 - 10 + 1 = 0$ ✓

$$y \cdot u_1 = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} = -15 + 45 + 5 = 35$$

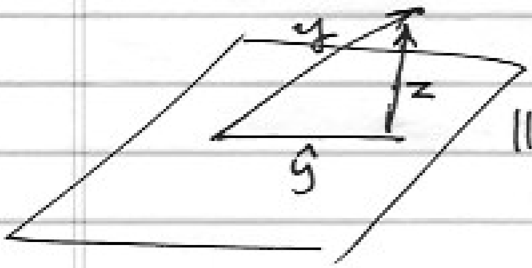
$$u_1 \cdot u_1 = 9 + 25 + 1 = 35$$

$$y \cdot u_2 = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = -15 - 18 + 5 = -28$$

$$u_2 \cdot u_2 = 9 + 4 + 1 = 14.$$

$$\hat{y} = \frac{35}{35} \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} + \frac{-28}{14} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}$$

$$z = y - \hat{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$



$$\|z\| = \text{distance} = \sqrt{2^2 + 0^2 + 6^2} = \sqrt{40}$$
$$= 2\sqrt{10}$$

Ex #17 $y = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$ $u_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$.

$\{u_1, u_2\}$ orthonormal

$$U = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \quad U^T = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$U^T U = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Thm 6 6.2

$$U \cdot U^T = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} \frac{8}{9} & \frac{-2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix}$$

$$\begin{bmatrix} \frac{8}{9} & \frac{-2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} = \text{proj}_W \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}.$$

Thm 10 If $\{u_1, u_2 .. u_p\}$ is an orthonormal basis for W.

Set $U = [u_1 \ u_2 \ .. \ u_p]$.

$$\text{proj}_W y = UU^T y$$