6.2 ORTHOGONAL SETS

Def $\vec{u}\cdot\vec{v}=0 \iff \vec{u}$ is orthogonal to $\vec{v}$

$\iff \vec{u}$ and $\vec{v}$ are orthogonal

Def A set $S=\{\vec{u}_1,\vec{u}_2,\dots,\vec{u}_p\}$ of vectors in $\mathbb{R}^n$ is called an orthogonal set, if $\vec{u}_i\cdot\vec{u}_j=0$ for $i\neq j$.

Ex 1 $\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix},\begin{bmatrix}-5\\-1\\2\end{bmatrix}\right\}$ is an orthogonal set since:

$$\begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}0\\2\\1\end{bmatrix}=0-2+2=0$$

$$\begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}-5\\-1\\2\end{bmatrix}=-5+1+4=0$$

$$\begin{bmatrix}0\\2\\1\end{bmatrix}\cdot\begin{bmatrix}-5\\-1\\2\end{bmatrix}=0-2+2=0.$$

Thm 4 If S is an orthogonal set of non-zero vectors, then
(i) S is independent set
(ii) S is a basis for its span.

Defn An orthogonal basis for subspace W is a basis which is an orthogonal set

Ex1 p1 is an orthogonal basis for $\mathbb{R}^3$.

Thm 5 Let W be a subspace of $\mathbb{R}^n$, and let $S = \{u_1, u_2 \dots u_p\}$ be an orthogonal basis for W. For each $y \in W$

$$y = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 + \dots + c_p \vec{u}_p \quad \text{where}$$

$$c_i = \frac{\vec{y} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i} \qquad \text{for } i = 1, 2, \dots, p$$

Example Exc #9

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}.$$

a) Check Orthogonal

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 0 - 2 = 0$$

$$\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -2 + 4 - 2 = 0$$

- Linearly independent since $\vec{u}_i \neq 0$ $i=1,2,3$.
- Spans $\mathbb{R}^3$, since 3 linearly independent vectors u_1, u_2, u_3, $\dim \operatorname{Span}\{u_1\ u_2\ u_3\} = 3$
  $\operatorname{span}\{u_1, u_2, u_3\} \subseteq \mathbb{R}^3$, $\dim \mathbb{R}^3 = 3$
  $\Rightarrow \operatorname{span}\{u_1, u_2, u_3\} = \mathbb{R}^3$.

Find c_1, c_2, c_3: $\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$$c_1 = \frac{\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} = \frac{8+0-3}{1+0+1} = \frac{5}{2}$$

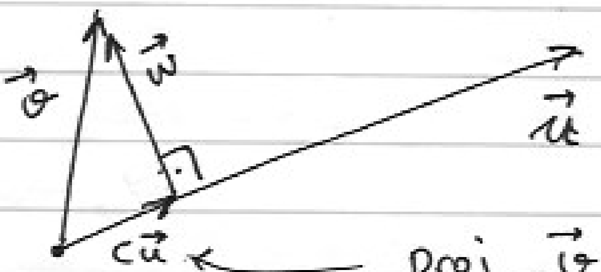
$$c_2 = \frac{\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}} = \frac{-8-16-3}{1+16+1} = \frac{-27}{18} = -\frac{3}{2}$$

$$c_3 = \frac{\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}} = \frac{16-4+6}{4+1+4} = \frac{18}{9} = 2$$

✓

Check. $$\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \stackrel{?}{=} \frac{5}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \left(-\frac{3}{2}\right) \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2}+\frac{3}{2}+4 \\ 0-6+2 \\ \frac{5}{2}-\frac{3}{2}-4 \end{bmatrix}$$

ORTHOGONAL PROJECTION



$\text{proj}_{\vec{u}}\vec{v}$ → parallel to $\vec{u}$ } want
→ $\vec{w} = \vec{v} - c\vec{u} \perp \vec{u}$

$$\text{proj}_{\vec{u}}\vec{v} = c\cdot\vec{u} \qquad (\text{parallel to } u)$$

$$c\vec{u}+\vec{w} = \vec{v}$$

$$\vec{u}\cdot(c\vec{u}+\vec{w}) = \vec{u}\cdot\vec{v}$$

$$c\cdot\vec{u}\cdot\vec{u} + \underbrace{\vec{u}\cdot\vec{w}}_{0} = \vec{u}\cdot\vec{v} \qquad \vec{w}\perp\vec{u}$$

$$c = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}$$

$$\boxed{\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}}$$

orthogonal projection of v onto $\vec{u}$

$$\boxed{w = \vec{v} - \frac{u\cdot v}{u\cdot u}u}$$

component of $\vec{v}$ orthogonal to $\vec{u}$.

Check $\vec{u}\cdot\vec{w} = \vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\cancel{\vec{u}\cdot\vec{u}}}\cancel{\vec{u}\cdot\vec{u}} = 0$

Exc #14 $\vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

Write $\vec{y} = \vec{v}_1 + \vec{v}_2$ s.t. $\vec{v}_1$ in Span(u)
$\vec{v}_2 \perp \vec{u}$.

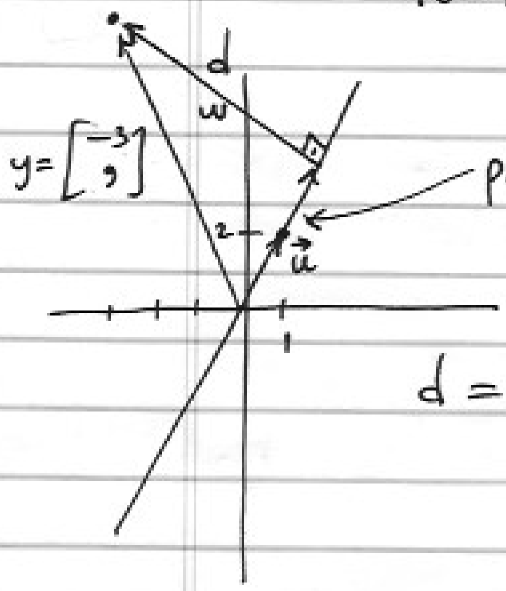
$$\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\, u = \text{proj}_u y = \frac{\begin{bmatrix} 2 \\ 6 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ 1 \end{bmatrix}}{\begin{bmatrix} 7 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ 1 \end{bmatrix}} \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$= \frac{14+6}{49+1}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \underset{\text{in span}(\vec{u})}{\vec{v}_1}$$

$$\vec{v}_2 = \vec{y} - \vec{v}_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}.$$

Check $\vec{v}_2 \cdot \vec{u} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ 1 \end{bmatrix} = -\frac{28}{5} + \frac{28}{5} = 0.$

Exc #16 Compute distance from $y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ to the line thru $\vec{u}$ & $\vec{0}$,
where $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



$$\text{proj}_u y = \frac{\begin{bmatrix} -3 \\ 9 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{-3+18}{1+4}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$d = \|w\| = \left\| \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -6 \\ 3 \end{bmatrix} \right\|$$

$$= \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$$

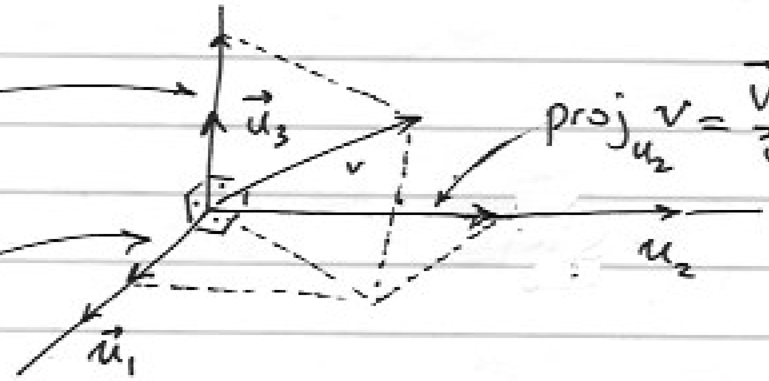
Recall Thm 5

$\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_p\}$ orthogonal basis for W

$v \in W: \quad \vec{v} = \sum_{i=1}^{p} \frac{\vec{v} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i} u_i$

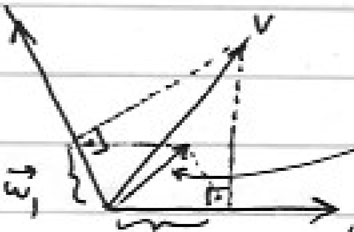
$\frac{\vec{v} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} \vec{u}_3 = \text{proj}_{u_3} v$

$\text{proj}_{u_2} v = \frac{\vec{v} \cdot \vec{u}_2}{\vec{u}_2 \cdot u_2} u_2$



$\frac{v \cdot u_1}{u_1 \cdot u_1} u_1 = \text{proj}_{u_1} v$

False if $\{u_1, u_2, \dots u_p\}$ is not orthogonal



$v \neq w_1 + w_2$

$\text{proj}_{u_1} v = \vec{w}_1$

w_2
$\shortparallel$
$\text{proj}_{u_2} v$

Defn A set $S = \{\vec{u}_1, \dots \vec{u}_p\}$ is called orthonormal if

(i) S is an orthogonal set, and

(ii) $\|\vec{u}_i\| = 1$, $i = 1, 2, \dots p$

Obs $\vec{u}_i \cdot \vec{u}_j = \begin{cases} 0 & \text{if } i \neq j \text{ (orthogonal)} \\ 1 & \text{if } i = j \text{ (unit)} \end{cases}$

for an orthonormal set $\{\vec{u}_1, \dots \vec{u}_p\}$

Consequently:

Theorem 6 Let $U = [u_1 \ u_2 \dots u_p]$, $m \times p$ matrix

$\underset{(p\times m)(m\times p)}{U^T U} = I_p \iff \{u_1, u_2 \dots u_p\}$ is orthonormal

Ex #19 $\begin{bmatrix} -.6 \\ .8 \end{bmatrix}, \begin{bmatrix} .8 \\ .6 \end{bmatrix}$ orthonormal? **YES**

$$\underbrace{\begin{bmatrix} -.6 & .8 \\ .8 & .6 \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} -.6 & .8 \\ .8 & .6 \end{bmatrix}}_{U} = \begin{bmatrix} .36 + .64 & -.48 + .48 \\ -.48 + .48 & .64 + .36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Exc #20 $\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$ $\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$.

Orthogonal? $\begin{bmatrix} -\frac{2}{3} \\ 1/3 \\ 2/3 \end{bmatrix} \cdot \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = -\frac{2}{9} + \frac{2}{9} = 0$ YES.

Orthonormal.?

$\left\| \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \right\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$ unit ✓

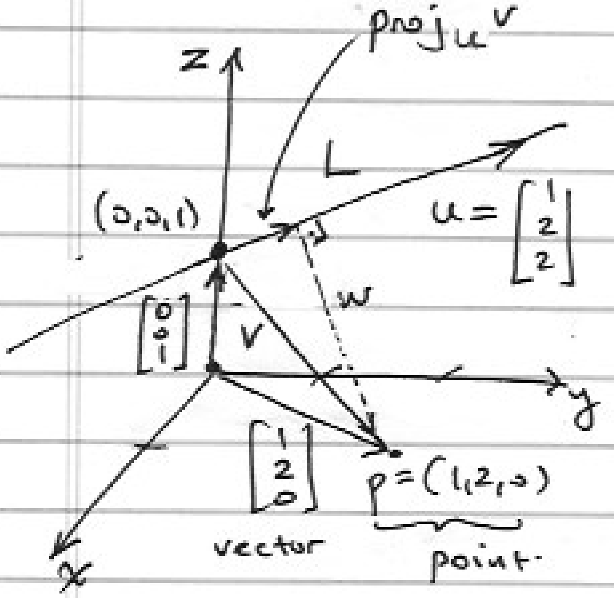
$\left\| \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} \right\| = \sqrt{\frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$ not unit.

$\left\{ \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} \right\}$ is not orthonormal

but

$\left\{ \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \frac{3}{\sqrt{5}} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} \right\}$ is orthonormal

Example



$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} + t\begin{bmatrix}1\\2\\2\end{bmatrix}$$

Line through (0,0,1) parallel to $\begin{bmatrix}1\\2\\2\end{bmatrix}$.

Distance between line L & $p=(1,2,0)$

$$v = \begin{bmatrix}1\\2\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix}.$$

$$\text{proj}_u v = \frac{\begin{bmatrix}1\\2\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\2\end{bmatrix}}{\begin{bmatrix}1\\2\\2\end{bmatrix}\cdot\begin{bmatrix}1\\2\\2\end{bmatrix}}\begin{bmatrix}1\\2\\2\end{bmatrix} = \frac{1+4-2}{1+4+4}\begin{bmatrix}1\\2\\2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}$$

$$= \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}.$$

$$w = v - \text{proj}_u v = \begin{bmatrix}1\\2\\-1\end{bmatrix} - \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}2/3\\4/3\\-5/3\end{bmatrix}$$

$$\|w\| = \sqrt{\frac{4}{9}+\frac{16}{9}+\frac{25}{9}} = \sqrt{5} = \text{distance between } p \text{ \& } L.$$