April 14, 2020

(Midterm 2 Review on April 16 lecture)

<u>Chap 6</u>

①

(6.1) Many new vocabulary. Make sure to learn all.

<u>Defn</u> Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be in $\mathbb{R}^n$.

Also known as dot product

$\vec{u} \cdot \vec{v}$, the inner product of $\vec{u}$ and $\vec{v}$ is defined to be:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

<u>Obs</u> If one looks at $\vec{u}$ and $\vec{v}$ as $n \times 1$ matrices:

$$u^T \cdot v = [\vec{u} \cdot \vec{v}] = [u_1\ u_2 \cdots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

<u>Ex</u> $\begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 2 \end{bmatrix} = 3 \cdot (-6) + 5 \cdot 2 = -8$

$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 1 \cdot 4 + (-1) \cdot 2 + 2 \cdot 0 = 2$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ not defined.

Properties of "$\cdot$" Dot product / Inner product

For all $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$:

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
- $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$ for all c in $\mathbb{R}$.
- $\vec{u} \cdot \vec{u} \geq 0$, and ($\vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0}$)

Question

we know

$$(AB)C = A(BC) \text{ for matrices.}$$

Is it true that

$$(\vec{u} \cdot \vec{v}) \vec{w} \overset{?}{=} \vec{u} (\vec{v} \cdot \vec{w}) \ ?$$

Ans No

$$\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) \begin{bmatrix} 5 \\ 6 \end{bmatrix} = (3+8) \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 55 \\ 66 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot (15+24) = \begin{bmatrix} 39 \\ 78 \end{bmatrix}$$

LENGTH If $\vec{v}$ is in $\mathbb{R}^n$, then the length (or norm) of $\vec{v}$ is defined to be $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

where $\vec{v} = (v_1, v_2, \ldots v_n)$.

Obs. $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

Example

$$\left\|\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}\right\| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$$

- A vector $\vec{u}$ is called unit if $\|\vec{u}\| = 1$.

Ex. $$\left\|\begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}\right\| = \sqrt{\left(\tfrac{1}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{-2}{3}\right)^2} = \sqrt{\tfrac{1}{9} + \tfrac{4}{9} + \tfrac{4}{9}} = 1$$

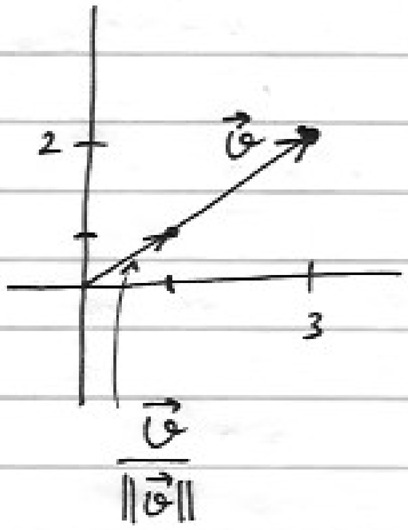
So $\begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$ is a unit vector.

- Normalization $\vec{v} \longrightarrow \dfrac{\vec{v}}{\|\vec{v}\|}$ if $\vec{v} \neq 0$

This gives a vector which is
(i) unit, and
(ii) in the same direction of the original.

Ex

Given $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Find a unit vector in the direction of $\vec{v}$.



$$\left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\| = \sqrt{4+9} = \sqrt{13}$$

$$\left\| \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix} \right\| = 1$$

answer:

$$\begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix}$$

normalization of v.

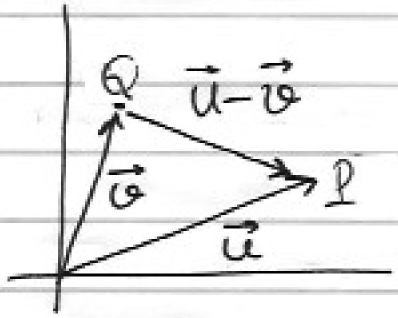
<u>Question</u> Is it always true that

$$\left\| \frac{v}{\|\vec{v}\|} \right\| = 1, \quad \text{if } \vec{v} \neq 0 \ ?$$ YES:

$$\left\| \frac{v}{\|v\|} \right\|^2 = \frac{v}{\|v\|} \cdot \frac{v}{\|v\|} = \frac{v \cdot v}{\|v\|^2} = \frac{\|v\|^2}{\|v\|^2} = 1.$$

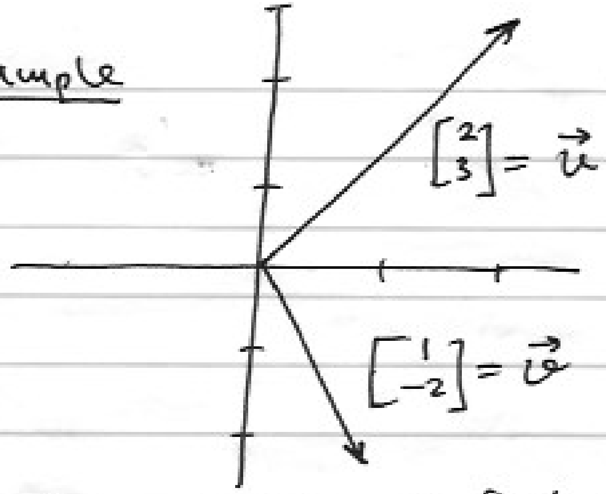
<u>Distance</u>

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \|\vec{v} - \vec{u}\|$$



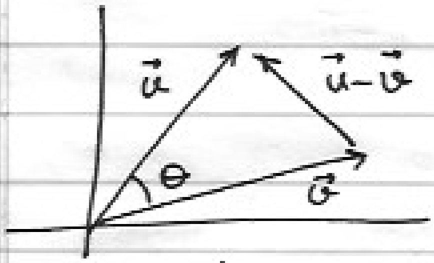
Distance between $\vec{u}$ and $\vec{v}$ actually means the distance between the terminal points (P and Q)

Example



$$\text{dist}\left(\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}\right) = \left\| \begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}1\\-2\end{bmatrix} \right\| = \left\| \begin{bmatrix}1\\5\end{bmatrix} \right\| = \sqrt{26}$$

Discussion



$$a = \|\vec{u}\|$$
$$b = \|\vec{v}\|$$
$$c = \|\vec{u}-\vec{v}\|$$

Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$.

$$c^2 = \|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$$
$$= a^2 + b^2 - 2\vec{u}\cdot\vec{v} = a^2 + b^2 - 2ab\cos\theta$$
$$\Rightarrow \quad \vec{u}\cdot\vec{v} = ab\cos\theta$$
$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

If $\vec{u},\vec{v} \neq 0$, $\quad \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|u\|\,\|v\|}$.

This formula extends to $\mathbb{R}^n$ since the triangle with vertices $\vec{0},\vec{u},\vec{v}$ is contained in a 2-plane, which is the same as $\mathbb{R}^2$ metrically.

The discussion on page 5 justifies

Def^n Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal to each other if $\vec{u}\cdot\vec{v} = 0$.

Def^n For two non-zero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right), \quad 0 \le \theta \le \pi$$

Observe that if $\vec{u}, \vec{v} \neq 0$:

$$\begin{aligned} \vec{u}\cdot\vec{v} = 0 &\iff \cos\theta = 0 \\ &\iff \theta = \frac{\pi}{2} \\ &\iff \vec{u} \perp \vec{v}. \end{aligned}$$

Ex Are $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix}$ orthogonal?

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix} = 3\cdot(-1) + 1\cdot(-5) + 4\cdot 2 = 0$$

Yes, orthogonal

Thm: (Pythagorean Thm)

$$\begin{aligned} \vec{u}\cdot\vec{v} = 0 &\iff \|u+v\|^2 = \|u\|^2 + \|v\|^2 \\ &\iff \|u-v\|^2 = \|u\|^2 + \|v\|^2 \end{aligned}$$

Why? put $u\cdot v = 0$ on p(5) discussion.

Ex. Find angle between $\begin{bmatrix}2\\2\\1\end{bmatrix}$ and $\begin{bmatrix}-3\sqrt{5}\\-\sqrt{7}\\2\sqrt{7}\end{bmatrix}$.

$$\left\|\begin{bmatrix}2\\2\\1\end{bmatrix}\right\| = \sqrt{4+4+1} = 3$$

$$\left\|\begin{bmatrix}-3\sqrt{5}\\-\sqrt{7}\\2\sqrt{7}\end{bmatrix}\right\| = \sqrt{45+7+28} = \sqrt{80} = 4\sqrt{5}$$

$$\begin{bmatrix}2\\2\\1\end{bmatrix}\cdot\begin{bmatrix}-3\sqrt{5}\\-\sqrt{7}\\2\sqrt{7}\end{bmatrix} = -6\sqrt{5} - 2\sqrt{7} + 2\sqrt{7} = -6\sqrt{5}$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{-6\sqrt{5}}{3\cdot 4\sqrt{5}} = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3} \quad (120^\circ)$$

Ex. Let $\vec{u} = \begin{bmatrix}-1\\2\end{bmatrix}$, $\vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$. Calculate $\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,u$

$$\vec{u}\cdot\vec{v} = \begin{bmatrix}-1\\2\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix} = -2+6 = 4$$

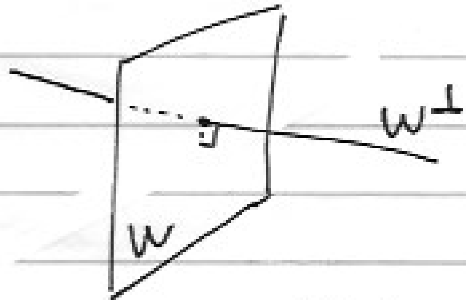
$$\vec{u}\cdot\vec{u} = \begin{bmatrix}-1\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\2\end{bmatrix} = 1+4 = 5$$

$$\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\cdot\vec{u} = \frac{4}{5}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-4/5\\8/5\end{bmatrix}.$$

ORTHOGONAL COMPLEMENTS

Def Given a subspace W of $\mathbb{R}^n$. The orthogonal complement $W^\perp$ (of W) consists all vectors $\vec{v}$ in $\mathbb{R}^n$ s.t. $\vec{v}\cdot\vec{w}=0$ for all vectors w in W.

Ex.



Observe that if $V = W^\perp$, then $V^\perp = W$.

Exc Let $u = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

W = the set of vectors $\vec{x}$ in $\mathbb{R}^3$ s.t. $\vec{u}\cdot\vec{x}=0$

Set $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad \vec{u}\cdot\vec{x} = x_1 - 2x_2 + 3x_3 = 0$

Rowreduce $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} = A$

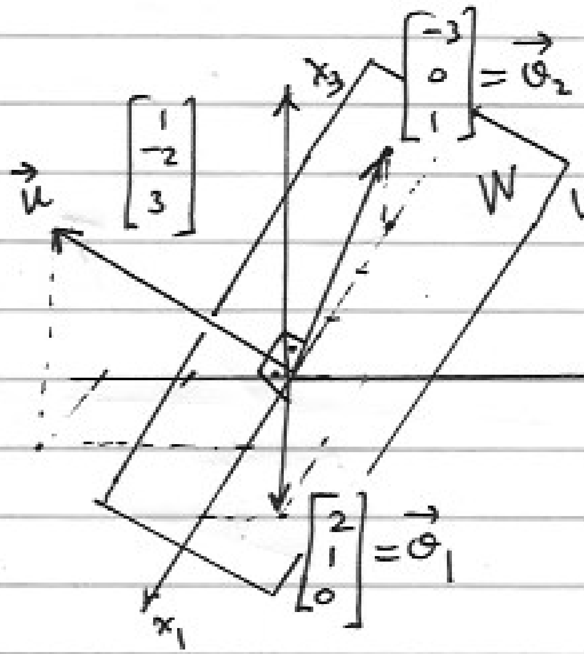
(1 is the pivot; free: x_2, x_3)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

$$W = \text{span}\left\{ \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2} \right\}$$

= Null space of A.



$W = \text{span}\{\vec{v}_1, \vec{v}_2\}$

$$v_1 \cdot v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = -6 \neq 0$$

$$u \cdot v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = -3 + 0 + 3 = 0$$

$$u \cdot v_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2 - 2 + 0 = 0$$

<u>Theorem</u>: Let A be an $m \times n$ matrix. Both row space of A (Row A) and null space of A (Nul A) are subspaces of $\mathbb{R}^n$; and

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and}$$

$$(\text{Nul } A)^\perp = \text{Row } A.$$