April 9, 2020

5.3 Continue

Example Diagonalize if possible:

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Solution: We are not given eigenvalues. so find them.

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 0 & -2+\lambda & 2-\lambda \end{vmatrix}$$

$-R_2+R_3 \to R_3$

factor $2-\lambda$

$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 3-\lambda & 2 & 1 \\ 1 & 4-\lambda & 1 \\ 0 & 0 & 1 \end{vmatrix}(2-\lambda)$$

$C_3+C_2 \to C_2$

$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)\big((3-\lambda)(4-\lambda)-2\big)$$

$$= (2-\lambda)\left(12-3\lambda-4\lambda+\lambda^2-2\right)$$
$$= (2-\lambda)(\lambda^2-7\lambda+10)$$
$$= (2-\lambda)(\lambda-5)(\lambda-2)$$

eigenvalues:

2, 2, 5

Find eigenspace for $\lambda = 2$

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 0$; free x_2, x_3.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

basis for eigenspace for 2

Find eigenspace for $\lambda = 5$

$$A - 5I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 + 2R_3} \begin{bmatrix} 0 & 3 & -3 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_3 = 0$
$x_2 - x_3 = 0$

free x_3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

basis for eigenspace 5

$$P = \begin{array}{c} \begin{array}{ccc} \lambda=2 & \lambda=2 & \lambda=5 \end{array} \\ \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Diagonalizable.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Check (i) Is P invertible? Yes since

$$\begin{vmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \underset{R_1+R_2}{=} \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} = 3 \neq 0$$

Check (ii) Is $A = PDP^{-1}$?

Suffices to check $AP = PD$.

$$AP = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 5 \\ 2 & 0 & 5 \\ 0 & 2 & 5 \end{bmatrix}$$

$$PD = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 5 \\ 2 & 0 & 5 \\ 0 & 2 & 5 \end{bmatrix}$$

This way you do not need to find P^{-1}.

The following theorems are true when we do all of our calculations, factorizations eigenvalues, eigenvectors and bases by using real numbers only (no complex numbers). In this course, we will not discuss diagonalization over complex numbers.

<u>Thm 6</u>: A $n \times n$ matrix with n distinct (real) eigenvalues is diagonalizable.

<u>Thm 7</u> Let A be an $n \times n$ matrix with distinct (real) eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_p$.

(a) For each $k = 1, 2, \cdots p$

$$1 \le \underbrace{\begin{pmatrix} \text{dimension of the} \\ \text{eigenspace for } \lambda_k \end{pmatrix}}_{\text{call } d_k.} \le \underbrace{\left(\begin{matrix} \text{multiplicity of } \lambda_k \\ \text{in the characteristic} \\ \text{polynomial} \end{matrix} \right.}_{\text{Call } m_k.}$$

(b) A is diagonalizable

$\iff d_1 + d_2 + \cdots + d_p = n$

$\iff$ (i) Characteristic polynomial of A factors into n linear factors (namely all roots are real), <u>and</u>
(ii) For all k $d_k = m_k$.

(c) If A is diagonalizable, and $\mathcal{B}_k$ is a basis for the eigenspace for λ_k, for all k, then combining all n vectors from all $\mathcal{B}_k$, one obtains an eigenvector basis for $\mathbb{R}^n$.

<u>Last example today</u>

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$\lambda_1 = 2 \quad m_1 = 2 \quad d_1 = 2$

$\lambda_2 = 5 \quad m_2 = 1 \quad d_2 = 1$

$d_1 + d_2 = 3 = n$
diagonalizable.

<u>Examples</u> from April 7

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

$\lambda_1 = -2 \quad m_1 = 1 \geq d_1 \geq 1$

$\lambda_2 = 5 \quad m_2 = 1 \geq d_2 \geq 1$

$CP = (\lambda - 5)(\lambda + 2)$ 2 distinct eigenvalues
$\Rightarrow$ diagonalizable

- $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} \longrightarrow CP = (\lambda - 3)^2.$

$\lambda_1 = 3 \quad m_1 = 2 > d_1 = 1$

not diagonalizable.

- $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda_1 = 0 \quad m_1 = 1 \geq d_1 \geq 1$

$\lambda_2 = 2 \quad m_2 = 1 \geq d_2 \geq 1$

$\lambda_3 = 3 \quad m_3 = 1 \geq d_3 \geq 1$

diagonalizable.

Exc #18 from April 7

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad 4\times 4$$

$$\text{Char. Poly} = (5-\lambda)^2(3-\lambda)(1-\lambda)$$

$\lambda_1 = 1 \qquad m_1 = 1 \geq d_1 \geq 1$

$\lambda_2 = 3 \qquad m_2 = 1 \geq d_2 \geq 1$

$\lambda_3 = 5 \qquad m_3 = 2 \geq d_3 \geq 1$

recall example

If $h \neq 6$ $\qquad d_3 = 1$

$d_1 + d_2 + d_3 = 3 \neq 4$

not diagonalizable

If $h = 6$ $\qquad d_3 = 2$

$d_1 + d_2 + d_3 = 4$

diagonalizable.

<u>Example</u> Diagonalize if possible

$$\begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{vmatrix} 0-\lambda & -1 & 0 \\ 4 & 0-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & -1 \\ 4 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2+4)$$

$\lambda_1 = 1$ only real eigenvalue

$0 = \lambda^2 + 4$ has no real roots.

$\lambda_1 = 1 \qquad m_1 = 1 \geqslant d_1 \geqslant 1$

$d_1 \neq n = 3.$

not diagonalizable over $\mathbb{R}$

Exc

$$A = \begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix} \quad \text{given } \lambda = 0, -1, -2$$

Is it diagonalizable?
If so find D, P s.t. $A = PDP^{-1}$.

Diagonalizable. 3 distinct real roots / eigenvalues for a 3×3 matrix. I can take

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

No short-cut for P.

$$A - \lambda I = A - 0 = \begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1]{RR} \begin{bmatrix} 1 & -1 & -1 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2-3R_1 \\ R_3-2R_1}]{} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

↑ free x_2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

← basis for eigenspace 0

Check

$$\begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

$\lambda = -1$

$$A - \lambda I = A + I = \begin{bmatrix} 3 & -2 & -2 \\ 3 & -2 & -2 \\ 2 & -2 & -1 \end{bmatrix} \xrightarrow[R_2 - R_1]{} \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 0 \\ 2 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow[R_1 - R_3]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 2 & -2 & -1 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \\ \uparrow \text{free} \end{array} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ \frac{1}{2} x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Basis for e-space -1

$$\text{Check} \quad \begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} \checkmark$$

$\lambda = -2$

$$A + 2I = \begin{bmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \xrightarrow[\frac{1}{2} R_3]{} \begin{bmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow[\substack{R_1 - 4R_3 \\ R_2 - 3R_3}]{} \begin{bmatrix} 0 & 2 & -2 \\ 0 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Check} \quad \begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

$$P = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \lambda=0 & \lambda=-1 & \lambda=-2 \end{array} \end{array} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$