April 7 2020

5.2 Continue & finish

Examples 1 Find characteristic polynomial & eigenvalues: $\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$.

$$\begin{vmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) - (-4)$$
$$= 21 - 7\lambda - 3\lambda + \lambda^2 + 4$$
$$= \lambda^2 - 10\lambda + 25 = (\lambda-5)^2$$

$$\lambda = \underbrace{5, 5}$$
(algebraic) multiplicity 2

Eigenspace for 5: $A - 5I = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

↑ free y

$x - y = 0$
$x = y$

Vector param. sol^n. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenspace for $\lambda = 5$: span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

dimension of eigenspace 5 is 1 even though 5 has multiplicity 2 in the characteristic polynomial.

Ex 2 Find Charateristic polynomial & eigenvalues $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix}$$

open along 3rd row

$$= (2-\lambda)\left((-1-\lambda)(4-\lambda) - 0 \right)$$
$$= (2-\lambda)(-4+\lambda-4\lambda-\lambda^2)$$
$$= (2-\lambda)(\lambda^2-3\lambda-4)$$
$$= 2\lambda^2-6\lambda-8-\lambda^3+3\lambda^2+4\lambda$$
$$= -\lambda^3+5\lambda^2-2\lambda-8$$

Characteristic polynomial simplified.

Eigenvalues $2, -1, 4$ from

Ex 3 #18 from the book:

It can be shown that for any eigenvalue λ:

$$\left(\begin{matrix}\text{algebraic} \\ \text{multiplicity}\end{matrix} \text{ of } \lambda\right) \geq \left(\begin{matrix}\text{dimension of the} \\ \text{eigenspace for } \lambda.\end{matrix}\right)$$

For

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Find h so that eigenspace for 5 is 2 dimensional.

$$\begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & h & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (5-\lambda)^2(3-\lambda)(1-\lambda)$$

eigenvalues : $\underbrace{5,5}_{}$, 3, 1.

algebraic multiplicity is 2

Eigenspace for $\lambda = 5$

$$A - 5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow[R_2 - R_1 \to R_2]{} \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & 0 & h-6 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} \xrightarrow[R_3 + R_4 \to R_4]{} \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & 0 & h-6 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{4}R_3]{-\frac{1}{2}R_1} \begin{bmatrix} 0 & 1 & -3 & \frac{1}{2} \\ 0 & 0 & h-6 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 0 & h-6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $h \neq 6$, then this matrix has rank 3,
nullity of $(A-5I)$ = dim of null space of $(A-5I)$
= (dim of eigenspace for 5) = 4 − 3 = 1.
by using Rank Thm Thm 14, section 2.9

If $h = 6$, then $\begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has rank 2

(dim of eigenspace for 5) = 4 − 2 = 2.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_3 \\ x_3 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

vector parametric sol^n for eigenspace 5

SIMILAR MATRICES

Defn Two $n \times n$ matrices A and B are called similar if there exists an invertible matrix P ($n \times n$) such that

$$A = PBP^{-1}$$

Basic properties ① "similarity" is a symmetric relation.

$$A = PBP^{-1} \implies A = Q^{-1} B Q.$$

Take $Q = P^{-1}$

$$QA = BQ$$
$$QAQ^{-1} = B.$$

② If A is similar to B then $\det A = \det B$.

$$A = PBP^{-1} \implies \det A = \det(PBP^{-1})$$
$$= \det P \cdot \det B \cdot \det P^{-1}$$
$$= \det B \cdot \det P \cdot \underbrace{\det P^{-1}}_{1/\det P}$$
$$= \det B.$$

③ If A is similar to B then

$$\det(A - \lambda I) = \det(B - \lambda I).$$

Consequently A and B have the same characteristic polynomials and same eigenvalues with same multiplicities.

Why? PTO

Given $A = PBP^{-1}$:

real #

$$
\begin{aligned}
P(B-\lambda I)P^{-1} &= PBP^{-1} - P(\lambda I)P^{-1} \\
&= PBP^{-1} - \lambda(PI)P^{-1} \\
&= A - \lambda \cdot PP^{-1} \\
&= A - \lambda I
\end{aligned}
$$

$$
\begin{aligned}
\det(A-\lambda I) &= \det\left(P(B-\lambda I)P^{-1}\right) \\
&= \det P \cdot \det(B-\lambda I) \cdot \det P^{-1} \\
&= \det(B-\lambda I) \cdot \underbrace{\det P \cdot \det P^{-1}}_{1} \\
&= \det(B-\lambda I).
\end{aligned}
$$

In the next section, 5.3, we will use similarities for diagonalization.

5.3 DIAGONALIZATION

<u>Defn</u> A square matrix A is called diagonalizable if it is similar to a diagonal matrix: that is there is an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

<u>Recall</u> A matrix is diagonal if $A_{ij} = 0$ for $i \neq j$

example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

- outside the main diagonal must be 0
- diagonal may have zeros.

<u>Example</u> $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^5 = \begin{bmatrix} 2^5 & 0 \\ 0 & 3^5 \end{bmatrix}$ <u>Diagonal</u>!

But

<u>Example</u> $\underbrace{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^2}_{A^2} = \underbrace{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}}_{A^2}$

compare.

$$A^2 \neq \begin{bmatrix} 2^2 & 3^2 \\ 4^2 & 5^2 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 16 & 25 \end{bmatrix} \neq A^2$$

For diagonalizable matrices there is a shortcut:

$$A = PDP^{-1}$$

$$A^2 = PD\underbrace{P^{-1}P}_{I}DP^{-1} = PD^2P^{-1}$$

$$A^k = PD\underbrace{P^{-1}\cdot P}_{I}D\underbrace{P^{-1}\cdot P}_{I}D\underbrace{P^{-1}\cdot P}_{I}D\underbrace{P^{-1}\cdots}_{I} \quad \underbrace{}_{I}P^{-1}DP^{-1}$$

$$A^k = PD^kP^{-1}.$$

<u>Example</u> Find $\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}^k$, if we are given

$$\underbrace{\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}}_{P^{-1}}$$

$$A^k = P\cdot D^k\cdot P^{-1}$$

$$= \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} (-3)^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot(-3)^k & (-2)(-2)^k \\ 2(-3)^k & -(-2)^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (-3)^{k+1} + (-2)^{k+2} & 6(-3)^k + 3(-2)^{k+1} \\ -2(-3)^k - (-2)^{k+1} & 4(-3)^k - 3(-2)^k \end{bmatrix}.$$

A is diagonalizable if $\begin{cases} A = PDP^{-1} \text{ for some} \\ \text{diagonal } D \text{ \& } \\ \text{invertible } P. \end{cases}$

***** Thm 5 (Diagonalization Thm) *****

Let A be an $n \times n$ matrix.

① A is diagonalizable if and only if A has n linearly independent eigenvectors

② $A = PDP^{-1}$ with a diagonal matrix D if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are the eigenvalues of A that correspond to columns of P, which are eigenvectors of A.

Ex. Given

$$\underset{A}{\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}} = \underset{P}{\begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}} \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{D} \underset{P^{-1}}{\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}}$$

inverses.

Find $\underbrace{\text{eigenvalues}}$, and basis for each eigenvalue.

Ans 5, 5, 4 from D. Basis for eigenspace for $\lambda = 5$ $\underbrace{\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}}_{\text{from } P}$; $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ basis for eigenspace $\lambda = 4$

Ex Diagonalize $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ if possible.

- $\begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)-12 = 2-3\lambda+\lambda^2-12$

$$= \lambda^2 - 3\lambda - 10$$
$$= (\lambda-5)(\lambda+2)$$

Eigenvalues : 5, −2

$\lambda = 5$ eigenspace

$$A-5I = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix}.$$

$x - \frac{3}{4}y = 0$

free parameter y

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{4}y \\ y \end{bmatrix} = y \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$$

basis for $\lambda = 5$ eigenspace

$\lambda = -2$ eigenspace

$$A+2I = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

free

$x + y = 0$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

basis for $\lambda = -2$ eigenspace

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3/4 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{4}{7} & \frac{4}{7} \\ -\frac{4}{7} & \frac{3}{7} \end{bmatrix}$$

$\lambda = 5$

calculate.

$$A = P \cdot D \cdot P^{-1}$$

$$\begin{bmatrix} 3/4 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{\frac{3}{4}+1} \begin{bmatrix} 1 & 1 \\ -1 & 3/4 \end{bmatrix}$$

Ex $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$ diagonalize if possible.

$$\begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0 \qquad \lambda = \underbrace{3,3}_{\text{double root.}}$$

$$A - 3I = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{free} & \text{pivot} \end{array}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Basis for eigenspace $\lambda = 3$: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

No other eigenvalue.

A is 2×2, there are no linearly independent 2 eigenvectors of A.
(All are multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.)

$$\underline{P} = \begin{bmatrix} 1 & ? \\ 0 & \end{bmatrix} \qquad D = \begin{bmatrix} \overset{\leftrightarrow \lambda}{3} & \\ 0 & ? \end{bmatrix}$$

A is <u>not</u> diagonalizable.