April 2, 2020

EIGENVALUES + EIGENVECTORS

OVERVIEW

Given a square matrix A, $n \times n$.

We want to find real numbers λ and vectors $v \neq 0$ satisfying

$$A \cdot v = \lambda \cdot v$$

$$A \cdot v = \lambda \cdot v \iff \underbrace{(A - \lambda I) v = 0}_{\substack{v=0 \text{ is always} \\ \text{a solution}}}$$

$(A - \lambda I) v = 0$ has a non-trivial soln

$\iff (A - \lambda I)$ is not invertible (Section 2.3 Thm 8)

$\iff \det(A - \lambda I) = 0$ (by Section 3.2 Thm 4)

1. If you want to find eigenvalues, solve $\det(A - \lambda I) = 0$. (5.2)

2 if you know a particular eigen value λ for A, then

- Find $A - \lambda I$.

- Row Reduce $A - \lambda I$

- Must have a free variable in $RREF(A-\lambda I)$ (otherwise λ is not an eigenvalue)

- Write vector parametric solution of $(A-\lambda I)\cdot x = 0$ as in 1.5

- Obtain a basis for the null-space of $A - \lambda I$ as in 2.8

Nullspace$(A-\lambda I)$ = Eigenspace of A for eigenvalue λ.

- Dimension of Eigenspace of A for λ
= # of parameters in sol^n of $(A-\lambda I)\cdot\vec{x} = 0$

= # vectors in a basis of the eigenspace for λ.

More examples from 5.1

Ex.

$$A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad \lambda = 4, \text{ eigenvalue}$$

Find a basis for the eigenspace for $\lambda = 4$.

$$A - 4I = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1+R_2 \to R_2]{} \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2+R_3 \to R_3]{} \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_1,\ -R_2]{} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free x_3 x_4

$$x_1 - 2x_3 = 0$$
$$x_2 - 3x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 3x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for Eigenspace for 4 $= \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

dim = 2

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ find one eigenvalue
no calculations

2 Identical rows $\Rightarrow \det A = 0$.

Always true: (for all square matrices)

$\det A = 0 \iff \det(A - 0 \cdot I) = 0$

$\iff 0$ is an eigenvalue.

Ex A 2×2 matrix can have at most two distinct eigenvalues. Why?

Thm: $\vec{v}_1, \dots \vec{v}_r$ eigenvectors to distinct eigenvalues $\lambda_1, \dots \lambda_r$ for A $(n \times n)$

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_r\}$ linearly independent.

But 3 vectors in $\mathbb{R}^2$ are linearly dependant. There cannot be 3 or more distinct eigenvalues.

Ex 2×2 matrix with one eigenvalue

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\lambda = 0$.

For any $\lambda \neq 0$ $\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}$

$= +\lambda^2 \neq 0$.

A cannot have a non-zero eigenvalue.

5.2

<u>Defn</u> For a given matrix A $(n\times n)$ $\det(A-\lambda I)=0$ is called the characteristic equation of A.

- Actually $\det(A-\lambda I)$ is a polynomial of degree n, if A is $n\times n$.

- The eigenvalues are the roots of $\det(A-\lambda I)=0$, and vice versa (meaning: all roots of $\det(A-\lambda I)=0$ are eigenvalues of A.)

Basic facts to remember from HS (or learn now if not seen before)

- All polynomials of degree n in $\mathbb{C}$ must have exactly n roots in $\mathbb{C}$, if counted with multiplicity. (Fundamental Thm of Algebra.)

<u>Ex</u> $(\lambda-5)^2(\lambda^2+1)=0 = \lambda^4-10\lambda^3+26\lambda^2-10\lambda+25.$

Roots $\lambda = \underbrace{5,5}_{\text{multiplicity }2}, i, -i$ 4 roots

- All polynomials of degree n in $\mathbb{R}$ must have at most n roots in $\mathbb{R}$.

- Let $P(\lambda)$ be a polynomial in $\mathbb{R}$ (or $\mathbb{C}$), where λ is the variable. Let a be a real number. Then

$$P(a)=0 \iff P(\lambda)=(\lambda-a)Q(\lambda)$$
for some polynomial $Q(\lambda)$

<u>Example</u> We want to find the roots of
$$P(\lambda)=\lambda^3-2\lambda^2-5\lambda+6=0.$$
Observe $P(1)=1-2-5+6=0$.
So, $\lambda^3-2\lambda^2-5\lambda+6=(\lambda-1)Q(\lambda)$
By long division, divide $\lambda^3-2\lambda^2-5\lambda+6$ by $\lambda-1$.
$$(\lambda^3-2\lambda^2-5\lambda+6)=(\lambda-1)\underbrace{(\lambda^2-\lambda-6)}$$
$$=(\lambda-1)(\lambda-3)(\lambda+2)$$
Roots of $P(\lambda)$ are $1, 3, -2$.

<u>Example</u> 1 (a) Find Characteristic Polynomial & eigenvalues for $A=\begin{bmatrix}-4 & -1\\ 6 & 1\end{bmatrix}$.

<u>Soln</u> $A-\lambda I=\begin{bmatrix}-4 & -1\\ 6 & 1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-4-\lambda & -1\\ 6 & 1-\lambda\end{bmatrix}$

$$\det A-\lambda I=\begin{vmatrix}-4-\lambda & -1\\ 6 & 1-\lambda\end{vmatrix}=(-4-\lambda)(1-\lambda)+6$$

$$=-4+4\lambda-\lambda+\lambda^2+6=\underbrace{\lambda^2+3\lambda+2}_{\text{Char. Poly.}}=(\lambda+1)(\lambda+2)$$
Eigenvalues: $-1, -2$.

(b) Let us find the eigenspaces:

$\lambda = -1$

$$A - \lambda I = A + I = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 6 & 2 \end{bmatrix}.$$

Row Reduce $\begin{bmatrix} -3 & -1 \\ 6 & 2 \end{bmatrix} \to \begin{bmatrix} -3 & -1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$

$x + \frac{1}{3}y = 0$ parameter y

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3\, y \\ y \end{bmatrix} = y \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

Basis for eigenspace for $\lambda = -1$.

$\lambda = -2$

$$A - \lambda I = A + 2I = \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$x + \frac{1}{2}y = 0$

free

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} y \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

basis for eigenspace for $\lambda = -2$

Ex Find characteristic polynomial & eigenvalues for

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$$

open along 2^{nd} row

$$\det A-\lambda I = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 5-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -2 & 7-\lambda \end{vmatrix}$$

$$\left.\begin{aligned} &= (5-\lambda)\left((3-\lambda)(7-\lambda)-(-2)1\right) \\ &= (5-\lambda)(21-7\lambda-3\lambda+\lambda^2+2) \\ &= (5-\lambda)(\lambda^2-10\lambda+23) \\ &= 5\lambda^2-50\lambda+115-\lambda^3+10\lambda^2-23\lambda \\ &= -\lambda^3+15\lambda^2-73\lambda+115 \end{aligned}\right\}\text{ charac. poly.}$$

Eigenvalues solve $0 = -\lambda^3+15\lambda^2-73\lambda+115$

Better use $0 = (5-\lambda)(\lambda^2-10\lambda+23)$

$\lambda = 5$

quadratic formula

$$\lambda = \frac{+10 \pm \sqrt{100-92}}{2}$$

$$= \frac{10 \pm \sqrt{8}}{2} = 5 \pm \sqrt{2}$$

Eigenvalues: $5,\ 5+\sqrt{2},\ 5-\sqrt{2}$ <u>all real</u>

Upper triangular / Lowertriangular / diagonal matrices

Ex $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 7 & 6 & 1 & 0 \\ -1 & 3 & 4 & -3 \end{bmatrix}$.

$$\det A - \lambda I = \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 5 & 2-\lambda & 0 & 0 \\ 7 & 6 & 1-\lambda & 0 \\ -1 & 3 & 4 & -3-\lambda \end{vmatrix}$$

open along first row

$$\det = (2-\lambda) \begin{vmatrix} 2-\lambda & 0 & 0 \\ 6 & 1-\lambda & 0 \\ 3 & 4 & -3-\lambda \end{vmatrix}$$

open along the first row

$$= (2-\lambda)(2-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 4 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(2-\lambda)(1-\lambda)(-3-\lambda)$$

eigenvalues $\underbrace{2, 2, 1, -3}$

they are the diagonal entries.

Example

Find characteristic polynomial & eigenvalues of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -3 & 3 & 2 \end{bmatrix}.$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 0-\lambda & 1 \\ -3 & 3 & 2-\lambda \end{vmatrix}$$ can be calculated directly to obtain

$$-\lambda^3+3\lambda^2-4.$$

Do you know how to solve a cubic eq^n, in general? Sometimes, if possible it is useful to keep factors, rather than multiplying out.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 0-\lambda & 1 \\ -3 & 3 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -\lambda-1 & 1 \\ -3 & 1+\lambda & 2-\lambda \end{vmatrix}$$

$C_2 - C_3 \downarrow C_2$ (column operations are allowed in determinants)

$(\lambda+1)$ factors in column 2

$$= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -1(\lambda+1) & 1 \\ -3 & 1(\lambda+1) & 2-\lambda \end{vmatrix} = (\lambda+1)\begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -1 & 1 \\ -3 & 1 & 2-\lambda \end{vmatrix}$$ add R_2 to R_3

$$= (\lambda+1)\begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 3-\lambda \end{vmatrix} = -(\lambda+1)\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix}$$

open along column 2

$$= -(\lambda+1)(\lambda^2-4\lambda+4)$$

e. values: $-1, \underbrace{2, 2}_{\text{double}}$.