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Chap V

5.1

EIGENVECTORS and EIGENVALUES

Defn An eigenvector of an $n \times n$ matrix A is a vector $\vec{v} \neq 0$ such that $A\vec{v} = \lambda \vec{v}$ for some real number $\lambda \in \mathbb{R}$. The number λ is called an eigenvalue of A, and $\vec{v} \neq 0$ is an eigenvector corresponding to λ, provided that $A\vec{v} = \lambda\vec{v}$.

Remarks: 1. $\lambda = 0$ is possible.
2. $\vec{v} \neq 0$ is required, since $A \cdot \vec{0} = \lambda \vec{0}$ for all A and λ.

Example Is 5 an eigenvalue of $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$?

We want to know if

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 5\begin{bmatrix} x \\ y \end{bmatrix}$$ has a non-trivial solution?

$$\left.\begin{aligned} 0 \cdot x + y &= 5x \\ 3x + 2y &= 5y \end{aligned}\right\} (*)$$

$$\begin{aligned} -5x + y &= 0 \\ 3x - 3y &= 0 \end{aligned}$$

$$\begin{bmatrix} -5 & 1 \\ 3 & -3 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 3 & -3 \\ -5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ -5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & -4 \end{bmatrix} \rightarrow$$

$$\to \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $x = y = 0$ is the only solution of (*).

5 is <u>NOT</u> an eigenvalue of $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$.

(b) Is 3 an eigenvalue of $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$?

<u>Solve</u> $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$

$$\left.\begin{aligned} 0x + y &= 3x \\ 3x + 2y &= 3y \end{aligned}\right\}$$

$$\begin{aligned} -3x + y &= 0 \\ 3x - y &= 0 \end{aligned}$$

$$\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 3 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x - \frac{1}{3}y = 0$$

Parametric sol^n $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}y \\ y \end{bmatrix} = y \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$. ← eigenvector

This will be a non-trivial solution for each $y \neq 0$

Check

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \underset{\text{eigenvalue}}{3} \underset{\text{eigenvector}}{\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}}$$

$$A \cdot v \qquad = 3 \cdot \vec{v}$$

Ex Is $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$?

Check $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-12+14 \\ 3-4+14 \\ 5-12+8 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix}$

Is there a λ s.t.

$$\lambda \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} ?$$

$$\left.\begin{aligned} \lambda &= 5 \\ -2\lambda &= 13 \\ 2\lambda &= 1 \end{aligned}\right\} \text{ no single } \lambda \text{ can satisfy all 3 equations.}$$

So $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ is NOT an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$.

Ex Is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$?

Check $\underbrace{\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{A \cdot \vec{v}} = \begin{bmatrix} -5+2 \\ -3+6 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \underbrace{3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{= 3 \cdot \vec{v}}$

Yes $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$ for the eigenvalue 3.

VERY IMPORTANT:

Let A be $n \times n$.

We are looking for $A \cdot \vec{v} = \lambda \vec{v}$ with $\vec{v} \neq 0$

$$A \cdot \vec{v} = \lambda \cdot I_n \vec{v}$$
$$A \cdot \vec{v} - \lambda I_n \vec{v} = 0$$
$$\underbrace{(A - \lambda I_n) \cdot \vec{v} = 0}$$

Is there a non-trivial solution?

Recall Thm 8 from 2.3

$B \cdot \vec{x} = 0$ has only the trivial solution

$\iff$ B is invertible

$\iff$ $\det B \neq 0$

So if we want $(A - \lambda I) \cdot \vec{v} = 0$ to have a solution $\vec{v} \neq 0$, then

$\underbrace{\det(A - \lambda I) = 0}$ must hold.

Called Characteristic Equation of A.

Helps us to find eigenvalues.

For a given eigenvalue λ, the solutions of the equation $(A - \lambda I) \cdot v = 0$ is the eigenspace of A corresponding to the eigenvalue λ.

Eigenspaces are subspaces of $\mathbb{R}^n$.

<u>Example</u>

$A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$. We are given $\lambda = 3, 7$ eigenvalues.

Find a basis for each eigenspace.

$\lambda = 3$

solve $\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

solve $\left(\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

solve $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free y $\quad x + y = 0$

$x = -y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

a basis for the eigenspace for $\lambda = 3$.

Check $\underbrace{\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{v} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \underbrace{3\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{= 3 \cdot v}$

PTO for $\lambda = 7$

$\lambda = 7$

Solve $\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 7\begin{bmatrix} x \\ y \end{bmatrix}$.

$$\left(\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

RReduce $\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$

$x - \frac{1}{3}y = 0$ free y

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}y \\ y \end{bmatrix} = y\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

$\left\{\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right\}$ basis for the eigenspace 7.

$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ basis for the eigenspace 3

$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right\}$ is linearly independent

Theorem: If $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_r$ are eigenvectors associated with distinct eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_r$, of an $n \times n$ matrix A, then

$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_r\}$ is linearly independent

Example

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix} \qquad \lambda = 3, 1 \text{ eigenvalues}$$

Find basis for eigenspaces.

$(A - \lambda I)v = 0$

$\lambda = 3$ $\quad A - 3I = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot

Free x_2, x_3

$x_1 - x_2 + x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Basis for the eigenspace for $\lambda = 3$

The eigenspace for $\lambda = 3$ is 2 dimensional

pto for $\lambda = 1$

$\lambda = 1$

$$A - 1I = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 3 \\ 2 & -2 & 4 \end{bmatrix}$$

$$\xrightarrow{RR} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -4 & 6 \\ 0 & -4 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 - 3R_1$

$R_3 - 2R_2$

free x_3

$$x_1 + \tfrac{1}{2} x_3 = 0$$
$$x_2 - \tfrac{3}{2} x_3 = 0$$

Vector Parametric Sol^n

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} x_3 \\ \frac{3}{2} x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix}$$

A Basis for eigenspace of $\lambda = 1$.

This is not the only basis.

Another basis $\left\{ \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} \right\}$

Another basis $\left\{ \begin{bmatrix} 5 \\ -15 \\ -10 \end{bmatrix} \right\}$

Ex

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda = 2$ is the only eigenvalue

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

free x_1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

A Basis for eigenspace 2 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

TRICK: For $\left.\begin{array}{l} \text{upper triangular} \\ \text{lower triangular} \\ \text{diagonal} \end{array}\right\}$ matrices,

the diagonal entries are the eigenvalues.

upper triangular $\begin{bmatrix} 1 & -1 & 6 \\ 0 & 2 & 100 \\ 0 & 0 & 5 \end{bmatrix} \longrightarrow \lambda = 1, 2, 5$ eigenvalues

lower triangular $\begin{bmatrix} 3 & 0 & 0 \\ 6 & 0 & 0 \\ 11 & 7 & 1 \end{bmatrix} \longrightarrow \lambda = 3, 0, 1$ eigenvalues

diagonal $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \longrightarrow \lambda = 2, 3$ eigenvalues.