More Examples on 3.2.

#8
$$\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix} \underset{\substack{-2R_1+R_2 \\ \downarrow \\ R_2}}{=} \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ -3 & -10 & -7 & 2 \end{vmatrix}$$

$$\xrightarrow[3R_1+R_4]{} \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -5 \\ 1 & 2 & 5 \\ -1 & -1 & -10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -5 \\ 0 & 0 & +10 \\ -1 & -1 & -10 \end{vmatrix}$$

Typo

open along 1st column

$R_2 - R_1 \to R_2$

$$= -10 \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = -10 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -10.$$

$R_1 + R_2$

#19 Ans 14 since $R_1 + 2R_2 \longrightarrow R_2$

#20 Ans 7 since $3R_3 + R_2 \longrightarrow R_2$

Prop Let A be an $n \times n$ (square) matrix.

$\det A \neq 0 \iff A^{-1}$ exists.

$\iff$ columns of A form a basis for $\mathbb{R}^n$

$\iff$ rows of A form a basis for $\mathbb{R}^n$

$\iff$ columns of A span $\mathbb{R}^n$

$\iff$ rows of A span $\mathbb{R}^n$

$\iff$ columns of A are lin indep

$\iff$ rows of A are lin indep.

Exc #21 $\begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{bmatrix}$ invertible? No; $\det = 0$

$$\det = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{vmatrix} \underset{R_3 - R_2 \to R_3}{=} \begin{vmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 2 & 6 & 0 \end{vmatrix} = -2 \begin{vmatrix} 2 & 6 \\ 2 & 6 \end{vmatrix} = 0.$$

$$\begin{vmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{vmatrix} \begin{matrix} \\ \\ \\ \end{matrix} = (12 + 0 + 36) - (0 + 36 + 12) = 0$$

$$\begin{matrix} 2 & 6 & 0 \\ 1 & 3 & 2 \end{matrix}$$

$$\left|\begin{array}{ccc|cc} 2 & 6 & 0 & 2 & 6 \\ 1 & 3 & 2 & 1 & 3 \\ 3 & 9 & 2 & 3 & 9 \end{array}\right| = (12 + 36 + 0) - (0 + 36 + 12)$$

$$= 0$$

#24 $\begin{bmatrix}4\\6\\2\end{bmatrix}\begin{bmatrix}-7\\0\\7\end{bmatrix}\begin{bmatrix}-3\\-5\\-2\end{bmatrix}$ linearly independent? NO

$$\begin{vmatrix}4 & -7 & -3\\6 & 0 & -5\\2 & 7 & -2\end{vmatrix} \underset{R_1+R_3 \to R_3}{=} \begin{vmatrix}4 & -7 & -3\\6 & 0 & -5\\6 & 0 & -5\end{vmatrix} = 0$$

two identical rows

#39 $\det A = -3$ 3×3
$\det B = 4$

$\det AB = \det A \cdot \det B = -12$

$\det 5A = \underset{3\times 3}{5^3} \cdot \det A = 125\cdot(-3) = -375$

$\det B^T = \det B = 4$

$\det A^{-1} = \dfrac{1}{\det A} = \dfrac{1}{-3} = -\dfrac{1}{3}$

$\det A^3 = (\det A)^3 = (-3)^3 = -27.$

3.3 Cramer's Rule

Ex

$$2x + 7y = 11$$
$$3x + 10y = 8$$

$$\begin{vmatrix} 2 & 7 \\ 3 & 10 \end{vmatrix} = -1 \neq 0$$

$$x = \frac{54}{-1} = -54$$

$$\begin{vmatrix} 11 & 7 \\ 8 & 10 \end{vmatrix} = 110 - 56 = 54$$

$$y = \frac{-17}{-1} = 17.$$

$$\begin{vmatrix} 2 & 11 \\ 3 & 8 \end{vmatrix} = 16 - 33 = -17.$$

Defn Let A be an $n \times n$ matrix, $\vec{b}$ be a column vector (in $\mathbb{R}^n$)

$$A_i(b) = \left[\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_{i-1} \ \vec{b} \ \vec{a}_{i+1} \cdots \ \vec{a}_n \right]$$

(In simple words: replaced the i^{th} column of A with $\vec{b}$.)

Theorem Cramer's Rule   Let $A\vec{x} = \vec{b}$ be given s.t. A is invertible ($\det A \neq 0$)

Then the unique solution of $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where} \quad x_i = \frac{\det A_i(b)}{\det A} \quad i = 1, 2, \ldots n.$$

Ex: Solve for x & y in terms of $s>0$.

$$\begin{cases} (e^s)x + (\cos s)y = s^2 \\ (s)x - (e^s)y = s+1 \end{cases}$$

$$\begin{vmatrix} e^s & \cos s \\ s & -e^s \end{vmatrix} = \underbrace{-e^{2s} - s\cos s}_{\text{as long as this is not } 0} = -(e^{2s} + s\cos s)$$

$$x = \frac{\begin{vmatrix} s^2 & \cos s \\ s+1 & -e^s \end{vmatrix}}{\begin{vmatrix} e^s & \cos s \\ s & -e^s \end{vmatrix}} = \frac{-s^2e^s - (s+1)\cos s}{-(e^{2s} + s\cos s)}$$

$$y = \frac{\begin{vmatrix} e^s & s^2 \\ s & s+1 \end{vmatrix}}{\begin{vmatrix} e^s & \cos s \\ s & -e^s \end{vmatrix}} = \frac{e^s(s+1) - s^3}{-(e^{2s} + s\cos s)}$$

More on Thursday for 3x3 examples.