3/5/2020

①

3.1 Continue

Thm: $\det A$ for an $n\times n$ matrix A can be calculated by using the cofactor expansion along any row or along any column:

$$\det A = \underbrace{\sum_{j=1}^{n} a_{ij}C_{ij}}_{\substack{i = \text{row } \# \\ i \text{ fixed}}} = \underbrace{\sum_{i=1}^{n} a_{ij}C_{ij}}_{\substack{j = \text{column } \# \\ j \text{ is fixed}}}$$

changing

i is changing

Ex

$$\begin{vmatrix} 2 & 3 & -1 \\ 4 & 1 & 6 \\ 5 & -2 & 4 \end{vmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \\ - & + & - \end{bmatrix}$$

$$\det A = \underset{a_{12}}{-\,3}\underset{A_{12}}{\begin{vmatrix} 4 & 6 \\ 5 & 4 \end{vmatrix}} + \underset{a_{22}}{1}\underset{A_{22}}{\begin{vmatrix} 2 & -1 \\ 5 & 4 \end{vmatrix}} - \underset{a_{32}}{(-2)}\underset{A_{32}}{\begin{vmatrix} 2 & -1 \\ 4 & 6 \end{vmatrix}}$$

$$= -3(16-30) + 1\cdot(8+5) + 2(12+4)$$

$$= 42 + 13 + 32 = 87.$$

Ex.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \end{bmatrix}$$

$$= -0 \begin{vmatrix} -2 & 5 & 2 \\ -6 & -7 & 5 \\ 0 & 4 & 4 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 & 2 \\ 2 & -7 & 5 \\ 5 & 4 & 4 \end{vmatrix} - 3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix} + 0 \begin{vmatrix} 1 & -2 & 5 \\ 2 & -6 & -7 \\ 5 & 0 & 4 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

$$= -3 \left[+5 \begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} - 0 \begin{vmatrix} \times \end{vmatrix} + 4 \begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix} \right]$$

$$= -3\big(5 \cdot (2) + 4(-2) \big) = -6.$$

Fast way for 2×2, 3×3

** No fast way for 4×4 or larger. Must expand

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + dhc + gbf) - (gec + fha + ibd)$$

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \\ \hline a & b & c \\ d & e & f \end{array}$$

$$\begin{array}{|ccc|} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{array} = (-24+0-50)-(-60+0-16)$$
$$\begin{array}{ccc} 1 & -2 & 2 \\ 2 & -6 & 5 \end{array}$$

$$= -74 + 76 = +2$$

3.1

upper triangular det.

$$\begin{vmatrix} 2 & 1 & 100 & 6 \\ 0 & -3 & 11 & e^{\pi} \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 6 \end{vmatrix} = 2(-3)(5)(6)$$

$$= -180$$ why?

$$= +6 \begin{vmatrix} 2 & 1 & 100 \\ 0 & -3 & 11 \\ 0 & 0 & 5 \end{vmatrix} + 0\text{'s}$$

$$= (+6)(+5)\begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} = (+6)(+5)(-3)\,|2|$$

$$= -180$$

1x1 det
not abs. value

Confusing notation, this is why people do not write 1x1 determinants

$|-5| = -5$

as 1x1 determinant

$|-5| = 5$

as absolute value

3.2

Effects of row operations on determinants

THM Let A be an $n \times n$ matrix.

- If two rows of A are interchanged to obtain a matrix B then
$$\det B = -\det A.$$

- If one row of A is multiplied with c to obtain B, then $\det B = c \cdot \det A$

- If a multiple of one row of A is added to another row of A $\underbrace{\text{(not multiplied)}}_{\text{second one}}$, to obtain B, then
$$\det A = \det B$$

see page 5 ***

Ex ① $\begin{vmatrix} \pi & e & e^6 \\ 0 & 0 & 0 \\ -1 & \ln 7 & 4 \end{vmatrix} = 0$

② $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 6 & 3 \\ 12 & 12 & 6 \end{vmatrix} = a$

$\quad ||$

$R_3 - 2R_2$ $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 6 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Ex

$$\begin{vmatrix} 3 & 6 & 9 \\ 1 & 1 & 3 \\ 4 & 7 & 6 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 4 & 7 & 6 \end{vmatrix} \xrightarrow{R_2 - R_1 \to R_2} = 3\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 4 & 7 & 6 \end{vmatrix}$$

$R_2 - R_1$

$$\underset{-4R_1 + R_3 \to R_3}{=} 3\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & -1 & -6 \end{vmatrix} \underset{R_3 - R_2 \to R_3}{=} 3\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -6 \end{vmatrix} = 3\cdot\left(1\cdot(-1)(-6)\right)$$

$$= 18$$

Ex

$$\underbrace{\begin{vmatrix} 1 & 2 \\ 5 & 8 \end{vmatrix}}_{-2} \xrightarrow[R_2 - 5R_1 \to R_2]{} \underbrace{\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}}_{-2}$$

$$\begin{vmatrix} 1 & 2 \\ 5 & 8 \end{vmatrix} \xrightarrow[R_2 - 5R_1 \to R_1]{} \begin{vmatrix} 0 & -2 \\ 5 & 8 \end{vmatrix} = 10 = (-2)(-5)$$

*** Caution $\quad cR_i + R_j \longrightarrow R_j$: det doesnot change.
replaces

$cR_i + R_j \longrightarrow R_i$: det gets multiplied with c.
replaces

Ex

$$\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} \underset{R_3 - 2R_4 \to R_3}{=\!=\!=} \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ -3 & 0 & -2 & 0 \\ 4 & 2 & 4 & 3 \end{vmatrix}$$

open along 4th column.

$$= 3 \begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ -3 & 0 & -2 \end{vmatrix} \underset{R_2 - 2R_1 \to R_2}{=} 3 \begin{vmatrix} -1 & 2 & 3 \\ 5 & 0 & -3 \\ -3 & 0 & -2 \end{vmatrix} = 3 \cdot 2 \, (-1) \begin{vmatrix} 5 & -3 \\ -3 & -2 \end{vmatrix}$$

$2 = a_{12}$

$$= (-6)(-10-9) = 19 \cdot 6 = 114.$$

THM Let A, B be $n \times n$ matrices

a) $\det A^T = \det A$

b) $\det A \cdot B = \det A \cdot \det B$

c) A^{-1} exists $\iff$ $\det A \neq 0$

d) A^{-1} exists $\implies$ $\det A^{-1} = \dfrac{1}{\det A}$

why?

$$A \cdot A^{-1} = I$$

$$\det A \cdot \det A^{-1} = \det I = 1$$

A, B given 4×4 matrices

$\det A = -1$

$\det B = +2$

$\det AB = (-1)(2) = -2$

$\det B^5 = 2^5 = 32$

$\det 2A = 2^4 \cdot -1 = -16$

$\det A^TA = \det A \cdot \det A^T = (-1)(-1) = +1$

$\det B^{-1}AB = \det B^{-1} \cdot \det A \cdot \det B$

$= \det A \cdot \underbrace{\det B \cdot \det B^{-1}}_{1}$

$\det B^{-1} = \frac{1}{2}$. $\det B = 2$

matrix multiplication does not commute, but real # multiplication commutes.