2.9 to Finish

Recall $Col(A)$ = column space of A
$nul(A)$ = Null space of A.

Defn. • $\dim col(A)$ = # vectors in any basis of $col(A)$
is called the Rank of A.

• $\dim nul(A)$ = # of vectors in any basis of $nul(A)$.
some texts call it nullity of A.

Rank Thm: If A is an $m \times n$ matrix,
then

$$\text{rank } A + \underbrace{\dim nul\, A}_{\text{nullity of } A} = n = \text{\# columns of } A.$$

PTO

Ex

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & -1 & 4 & -7 \\ -2 & -1 & 3 & -7 & 6 \\ 3 & 4 & -2 & 7 & 9 \end{bmatrix} \to\to \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & -1 & 3 & -4 \\ 0 & 3 & 3 & -9 & 12 \\ 0 & -2 & -2 & 10 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

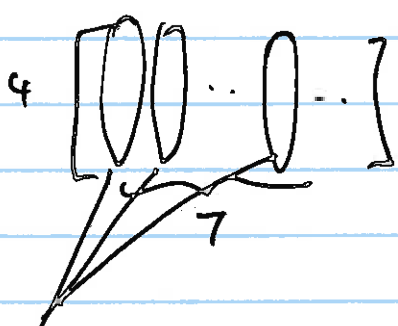
$$\to\to \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 3

nullity = 2

\# columns = 5

2.9 #16 4×7, 3 pivot columns: Col $A = \mathbb{R}^3$? dim null(A) = ?



$$\to \begin{bmatrix} 1^* & 0 & 2 & 1 & 0 & 1 & 1 \\ 0 & 1^* & 3 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1^* & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Yes, col space has 3 basis vectors spans a subspace $\sim \mathbb{R}^3$, but not $\mathbb{R}^3$ due to moving occured via row reduction.

$7 = \text{nullity} + \text{rank} = 4 + 3$

29 #20 4×5 matrix

null space 3 dim

$\Rightarrow$ rank $= 2$

Obs

$$\operatorname{rank}(\underset{m \times n}{A}) \leq \min(m, n)$$

Ex rank of (3×7 matrix A) ≤ 3

rank of (7×3 matrix A) ≤ 3

End of Chap II.

Chap III

3.1

Determinants

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$ad-bc = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Higher determinants are done by using reduction to smaller size determinants

$$n\times n \longrightarrow (n-1)\times(n-1) \longrightarrow (n-2)\times(n-2) \cdots \rightarrow 2\times 2 \rightarrow 1\times 1$$

know

Given a matrix (Square!)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & & & a_{nn} \end{bmatrix}$$

a_{ij} = i^{th} row, j^{th} column entry of A

$\neq$

A_{ij} is obtained by removing all of the i^{th} row
by " all of the j^{th} column

$a_{23} = 3$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 3 \\ 6 & 1 & 0 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} 1 & 2 \\ 6 & 1 \end{bmatrix} \qquad A_{32} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$$

Defn $C_{ij} = (-1)^{i+j} \det A_{ij}$ cofactor

$$(-1)^{i+j} : \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \\ + & - & + & - \end{bmatrix}$$

Defn Given an $n\times n$ matrix A, $n \geq 2$

we define

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}$$

$$= a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

$$\left[= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - \cdots \right.$$

$$\cdots + (-1)^{1+j} a_{1j} \det A_{1j} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}.$$

Ex $\begin{vmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \\ -2 & 0 & 6 \end{vmatrix}$.

$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

$a_{11} = 2$ $\qquad A_{11} = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$

$a_{12} = -1$ $\qquad A_{12} = \begin{bmatrix} 1 & 2 \\ -2 & 6 \end{bmatrix}$

$a_{13} = 0$ $\qquad A_{13} = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$

$C_{11} = + \det A_{11} = 6$

$C_{12} = - \det A_{12} = -10$

$C_{13} = + \det A_{13} = 2$

$\det A = 2 \cdot 6 + (-1)(-10) + 0 \cdot 2 = 12 + 10$

$= 22.$

Ex 2

$$\begin{vmatrix} -1 & 4 & 2 \\ 0 & 1 & 5 \\ 3 & 2 & -1 \end{vmatrix}$$

$a_{11} = -1 \qquad A_{11} = \begin{bmatrix} 1 & 5 \\ 2 & -1 \end{bmatrix} \xrightarrow{\det} -11 \qquad C_{11} = -11$

$a_{12} = 4 \qquad A_{12} = \begin{bmatrix} 0 & 5 \\ 3 & -1 \end{bmatrix} \xrightarrow{\det} -15 \qquad C_{12} = 15$

$a_{13} = 2 \qquad A_{13} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \xrightarrow{\det} -3 \qquad C_{13} = -3$

$$\det \quad (-1)(-11) + 4 \cdot 15 + 2 \cdot (-3)$$

$$= 11 + 60 - 6 = 65$$