Feb 25, 2020

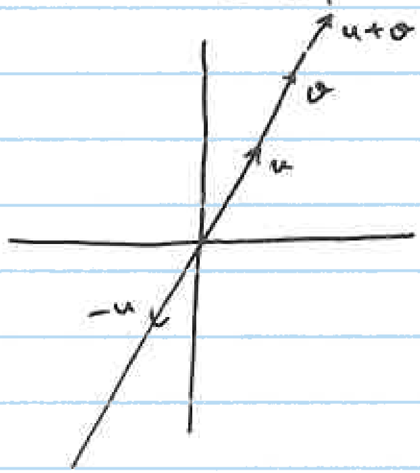
①

2.8 Subspaces, Col(A), Nul(A)

Def: A set $H \subseteq \mathbb{R}^n$ is called a subspace if

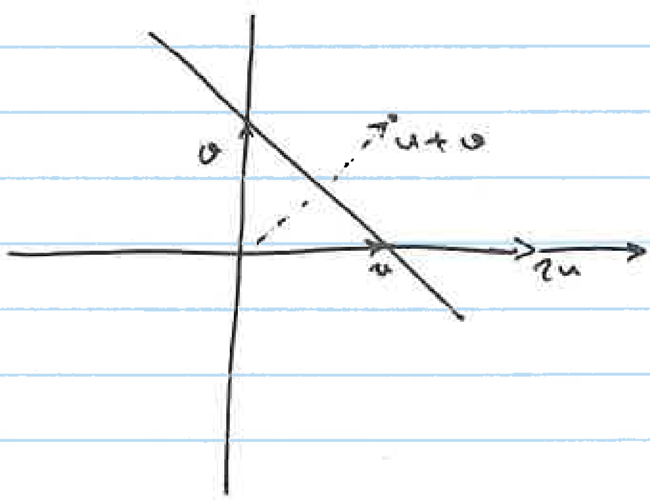
- $\vec{u}, \vec{v} \in H \Rightarrow \vec{u} + \vec{v} \in H$, and
- $\vec{u} \in H$, $c \in \mathbb{R} \Rightarrow c \cdot \vec{u} \in H$.

Ex ① $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, y = 2x \right\} \subseteq \mathbb{R}^2$

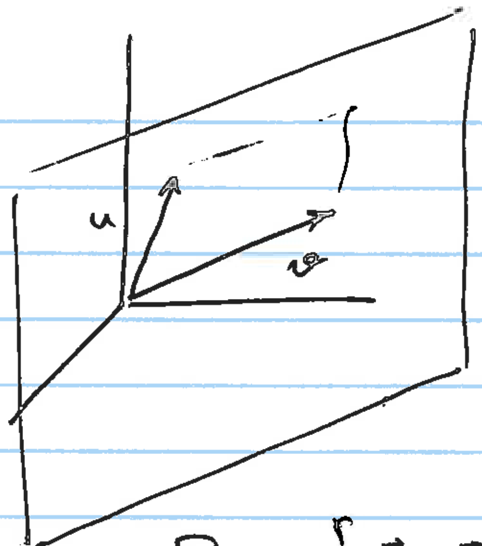


Subspace

② $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, x + y = 1 \right\}$



not a subspace



In $\mathbb{R}^3$

Take $\vec{u}, \vec{v}$, both non-zero

not parallel,

$$\text{Span}\{\vec{u}, \vec{v}\} = \text{2-plane thru origin}$$
& tips of $\vec{u}$ & $\vec{v}$

<u>Prop</u> For any given set of vectors $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_p\}$

in $\mathbb{R}^n$, the span $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_p\}$

$$= \{ c_1\vec{w}_1 + c_2\vec{w}_2 + \dots + c_p\vec{w}_p \mid c_1, c_2, \dots c_p \in \mathbb{R} \}$$

= the set of all possible linear combinations of $\vec{w}_1, \vec{w}_2, \dots \vec{w}_p$

is a subspace of $\mathbb{R}^n$.

<u>In $\mathbb{R}^3$</u> The subspaces are:

- $\{\vec{0}\}$
- lines thru origin
- planes thru origin
- $\mathbb{R}^3$

Defn A subset B of a subspace H is called a basis if

- B is linearly independant
- B spans H.

Consequence of this:

- every vector in H is a linear combination of vectors from B, and
- every vector in H can be represented as a linear combination from B in a unique way.

Ex

$$\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$$

is the xy plane in x, y, z space.

$$\begin{bmatrix}3\\5\\0\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}3\\5\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Why can I do this in different ways?

Ans: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ is <u>not</u> linearly independant

If $H = \mathbb{R}^n$, $B = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$.

$$A = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \end{bmatrix}}_{\text{as column vectors}} \xrightarrow{RR} RREF(A)$$

$\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$ spans $\mathbb{R}^n$

$\iff$ Every row of RREF of A has a leading 1 of a row.

$\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$ linearly independent $\iff$ Every column of RREF(A) has a leading 1 of a rows

$\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$ is a basis of $\mathbb{R}^n$ $\iff$ $RREF(A) = I_n$.

Example

a

Given $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix} \right\}$

Is B a basis of $\mathbb{R}^3$?

correction

$$\begin{bmatrix} 1 & 3 & 5 \\ 1 & -1 & 1 \\ -3 & 2 & -4 \end{bmatrix} \longrightarrow \longrightarrow \longrightarrow \begin{bmatrix} 1^* & 0 & 2 \\ 0 & 1^* & 1 \\ 0 & 0 & 0 \end{bmatrix} = RREF$$

not basis for $\mathbb{R}^3$

not spanning $\mathbb{R}^3$

not lin independent

b Another example

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 5 \end{bmatrix} \xrightarrow{RREF} \longrightarrow \longrightarrow \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

Basis for $\mathbb{R}^3$	No
Span $\mathbb{R}^3$	yes
lin Indp	No

IMPORTANT *****

COLUMN SPACE

Given an $m \times n$ matrix A, the subspace of $\mathbb{R}^m$ spanned by the columns of A is called the column space of A denoted by $\mathrm{COL}(A)$.

NULL SPACE

Given an $m \times n$ matrix A the subspace of $\mathbb{R}^n$ defined by $\{\vec{x} \in \mathbb{R}^n \mid A \cdot \vec{x} = 0\}$ is called the null-space of A, denoted by $\mathrm{NUL}(A)$

method 1 Do RREF

2 see next page.

Ex $A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix}$ 3×4

Find $\mathrm{COL}(A)$, $\mathrm{NUL}(A)$

$$\begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \xrightarrow[R_1 - R_2 \to R_1]{} \begin{bmatrix} 1 & -2 & 2 & -2 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

NULL SPACE : $A \cdot \vec{x} = 0$

$$\begin{bmatrix} 1 & -2 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot pivot

free parameters x_2, x_4

$$x_1 - 2x_2 - 6x_4 = 0$$
$$x_3 + 2x_4 = 0$$

$x_1 = 2x_2 + 6x_4$

$x_2 = x_2$ free

$x_3 = -2x_4$

$x_4 = x_4$ free

Write soln in vector parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

basis for NUL(A)

NULL SPACE = Span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

in $\mathbb{R}^4$

A is 3x4 matrix.

COLUMN SPACE :

Span $\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix} \right\}$

in $\mathbb{R}^3$

<u>Why</u>: column 1, column 3 of RREF(A) are pivot.

Take column 1, column 3 of A

to get COLUMN space basis.

ADDITIONAL EXAMPLE (As promised in class).
Find bases for col A and nul A where
$$A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 4 & 6 \end{bmatrix}.$$

Sol^n Reduce $\begin{bmatrix} 2 & 3 & 2 \\ 1 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 6 \\ 2 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 6 \\ 0 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

$$\downarrow$$

$$\begin{array}{ccc} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix} \\ \uparrow \ \uparrow \quad \uparrow \\ \text{pivots} \quad \text{free} \end{array}$$

- $A\vec{x} = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Basis for null space $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

Basis for column space pivot columns of RREF(A) are #1, #2 columns.

Go to A #1, #2 columns

$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ basis for col A.