2.2 To finish 2.3 To start

Properties of inverses

- $(A^{-1})^{-1} = A$

- $(A \cdot B)^{-1} = B^{-1} A^{-1}$

  If both A and B are invertible (which requires square matrices)

- $(A^T)^{-1} = (A^{-1})^T$.

2nd identity

Why?

$$(A \cdot B)(B^{-1} \cdot A^{-1}) = A \cdot \underbrace{(B \cdot B^{-1})}_{I} \cdot A^{-1}$$

$$= \underbrace{A \cdot I}_{A} \cdot A^{-1}$$

$$= A \cdot A^{-1} = I.$$

$$(A \cdot B) \cdot (B^{-1} \cdot A^{-1}) = I$$

& similarly

$$(B^{-1} \cdot A^{-1}) \cdot (A \cdot B) = I$$

$$\Longrightarrow (A \cdot B)^{-1} = B^{-1} A^{-1}$$

CAUTION Example

$$\begin{bmatrix} 2 & 1 & 0 \\ 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2 \times 3 \qquad 3 \times 2 \qquad 2 \times 2$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 5 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 7 & 0 \end{bmatrix}$$

3x2 2x3 3x3

$$\begin{bmatrix} 2 & 1 & 0 \\ 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \cdot B = I_2$$

$$B^{-1} \cdot A^{-1} \neq (A \cdot B)^{-1} = I_2$$

Don't exist
not square
matrices.

2.3 Characterization of Invertible matrices

Ex

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & +5 \end{bmatrix}$$

2×3

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - y \\ 3x + 4y + 5z \end{bmatrix}$$

$\mathbb{R}^3$ $\quad$ $\mathbb{R}^2$

This defines $L : \mathbb{R}^3 \to \mathbb{R}^2$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ 3x + 4y + 5z \end{bmatrix}$$

Called a linear map
linear function
linear transformation

Def^n Let A be an $m \times n$ matrix.
One can define $L : \mathbb{R}^n \to \mathbb{R}^m$
by
$$L(\vec{x}) = A \cdot \vec{x},$$
this is the linear map associated with the matrix A.

Observe that if A is $n \times n$
and if A is invertible

then $L(x) = A \cdot x \; : \; \mathbb{R}^n \to \mathbb{R}^m$

$M(x) = A^{-1} \cdot x \; : \; \mathbb{R}^n \to \mathbb{R}^n$

$$(L \circ M)(\vec{x}) = A \cdot (A^{-1} \cdot x) = x$$

$$(M \circ L)(\vec{x}) = A^{-1} \cdot (A \cdot x) = x$$

A is invertible matrix. $\iff$ $L(x) = A \cdot x$ is an invertible function

MUST KNOW

Thm 8 12 equivalent conditions (See the textbook) for SQUARE matrices

summarizes many facts we learned earlier.

- Thm 8 is not true for non-square matrices
- We need to learn how to use it to solve many different type of problems.

Ex $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ columns are linearly independent in $\mathbb{R}^3$ but columns do not span $\mathbb{R}^3$

From Thm 8: (e) is true
(h) is false

How come?
since Thm 8 Does not apply to non-square matrices.

Ex $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ Columns span $\mathbb{R}^2$ but A is not invertible

From Thm 8 (a) false
(h) is true

same.

Exc #4 $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$ is not invertible since RREF(A) will have all 0 in 2^{nd} column.
leading 1's $\leq 2 < 3$

Exc 8
Exc 13

* Upper triangular and Main diagonal all $\neq 0$ $\Longrightarrow$ Invertible

Do NOT use for not triangular / not diagonal matrices

2.3 #18 If C is 6×6,

$C\vec{x} = \vec{v}$ is consistent for every $\vec{v}$ in $\mathbb{R}^6$, then

Is it possible to have for some $\vec{v}$,

the equation $C\vec{x} = \vec{v}$ has more than one solution? NO

Thm 8 (g) holds

(g) $\Rightarrow$ (c) holds $\longrightarrow$ n pivot positions in $n \times n$ matrix

(g) $\Rightarrow$ (f) holds

$\downarrow$

$x \to Ax$ is 1-1 $\longrightarrow$ unique sol^n for each $\vec{v}$

(g) $\Longrightarrow$ (a) holds

$C\vec{x} = \vec{v}$

$\vec{x} = C^{-1}\vec{v}$