# 2.2 INVERSE of A MATRIX

Properties of Matrix Multiplication:

$A(BC) = (AB)C$ Associative

$A(B+C) = AB + AC$ Distributive (from left)

$(B+C)A = BA + CA$ " (from right)

$(rA)B = r(AB)$ Associative (with scalar multiplication)

If $A$ is $m \times n$

$$I_m A = A = A \cdot I_n$$

Ex

$$\underset{\substack{I_2 \\ 2\times 2}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 6 \end{bmatrix}} \underset{\substack{I_3 \\ 3\times 3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 6 \end{bmatrix}$$

CAUTION: $\underbrace{AB \neq BA \text{ in general}}$

$AB \neq BA$ in some cases

$AB = BA$ in some other cases

- $AB = AC \not\Rightarrow B = C$ OR $A = 0$
- $AB = 0 \not\Rightarrow A = 0$ OR $B = 0$

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

<u>Defn</u> Given an $n \times n$ matrix $A$

IF there is a matrix $C$ $(n \times n)$ s.t.

$$A \cdot C = I_n$$

and

$$C \cdot A = I_n,$$

then we call $A$ invertible & $C = \underbrace{A^{-1}}_{\text{inverse of } A.}$

CAUTION

- Not every square matrix is invertible
  Some are, some aren't
- Non-square matrices are <u>never</u> invertible

Ex

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.$

Ex

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{0}$$

$A$ can't have an inverse! Why?

Suppose I have $A^{-1}$: (Something will go wrong!

$$A \cdot B = 0 \quad \text{above}$$

$$A^{-1} \cdot (A \cdot B) = 0 = A^{-1} \cdot 0$$

$$\underbrace{(A^{-1} \cdot A)}_{I_n} \cdot B = 0$$

$$I_n \cdot B = 0$$

$$B = 0$$

But $B \neq 0$
$$B = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$$

Contradiction.

Conclusion: $A^{-1}$ doesn't exist.

## Common terminology

**Def$^n$** $A^{-1}$ exists $\iff$ $A$ is invertible

$\iff$ $A$ is non-singular

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How do I find $A^{-1}$, if it exists?

(1x1) $[2]^{-1} = [\frac{1}{2}]$

$[2][\frac{1}{2}] = [1] = I_1$

$[a]^{-1} = [\frac{1}{a}]$ if and only if $a \neq 0$

$[0]^{-1}$ : Doesn't exist.

(2x2) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any 2x2 matrix.

$\Delta = ad - bc$ (determinant)

If $\Delta = 0$, then $A^{-1}$ doesn't exist.

If $\Delta \neq 0$ $A^{-1} = \begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}$.

Let us check.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{ad}{\Delta} - \frac{bc}{\Delta} & \underbrace{\frac{-ab}{\Delta} + \frac{ba}{\Delta}}_{0} \\ \underbrace{\frac{dc}{\Delta} - \frac{cd}{\Delta}}_{0} & \frac{-bc}{\Delta} + \frac{ad}{\Delta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\frac{-bc}{\Delta} + \frac{ad}{\Delta} = \frac{ad-bc}{\Delta} = 1$$

since $\Delta = ad - bc$

$$\begin{bmatrix} 2 & 3 \\ -6 & 7 \end{bmatrix}^{-1} = \frac{1}{32}\begin{bmatrix} 7 & -3 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{32} & \frac{-3}{32} \\ \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

$$\Delta = 2 \cdot 7 - (-6 \cdot 3) = 14 + 18 = 32$$

# ALGORITHM TO FIND $A^{-1}$ (For all $n=1,2,3\ldots$

Given $A$

- If $A$ is not square, $A^{-1}$ DNE.
  does not exist
- If $A$ is square:

$$n\left\{\underbrace{\left[\begin{array}{c:c} A & I \end{array}\right]}_{2n}\right. \xrightarrow{\text{both sides}} \longrightarrow$$

Row Reduce whole $n\times 2n$ matrix both sides simultaneously

$$\left[\begin{array}{c:c} \text{RREF}(A) & C \end{array}\right].$$

Case 1 $\text{RREF}(A)=I_n$ then $C=A^{-1}$.

<u>Case 2</u> $\text{RREF}(A)\neq I_n$ then $A^{-1}$ <u>DNE</u>!

does not exist

Ex. a

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 5 \end{bmatrix}$$

Does $A^{-1}$ exist? If so, what is it?

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[4R_1+R_2 \to R_2]{} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ -2 & -6 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_1+R_3 \to R_3]{} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1 \to R_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 0 & 1 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 3 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_2+R_3 \to R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 0 & 1 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 10 & 2 & 1 \end{array}\right] \xrightarrow[R_3+R_2 \to R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 14 & 3 & 1 \\ 0 & 0 & 1 & 10 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[R_1-2R_3 \to R_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 & -4 & -1 \\ 0 & 1 & 0 & 14 & 3 & 1 \\ 0 & 0 & 1 & 10 & 2 & 1 \end{array}\right]$$

$$I_3 \implies A^{-1}.$$

Check

$$\begin{bmatrix} -17 & -4 & -1 \\ 14 & 3 & 1 \\ 10 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

Solve

$$\left.\begin{aligned} x+2y-z&=0\\ -4x-7y+3z&=11\\ -2x-6y+5z&=4 \end{aligned}\right\} (*)$$

and

$$\left.\begin{aligned} x+2y-z&=1\\ -4x-7y+3z&=3\\ -2x-6y+5z&=0 \end{aligned}\right\} (**)$$

Method Solve $A\vec{x}=\vec{b}$, if $A^{-1}$ exist!

$$A^{-1}\cdot(A\vec{x}) = A^{-1}\cdot b$$

$$\underbrace{(A^{-1}\cdot A)}_{I}\cdot\vec{x} = A^{-1}b$$

$$\vec{x} = A^{-1}b$$

If b is given:
In this case
Always unique sol$^{n}$

$$\begin{bmatrix}1 & 2 & -1\\ -4 & -7 & 3\\ -2 & -6 & 5\end{bmatrix}^{-1} = \begin{bmatrix}-17 & -4 & -1\\ 14 & 3 & 1\\ 10 & 2 & 1\end{bmatrix}$$

Know from previous page.

Sol$^{n}$ (*)

$$\begin{bmatrix}-17 & -4 & -1\\ 14 & 3 & 1\\ 10 & 2 & 1\end{bmatrix}\begin{bmatrix}0\\ 11\\ 4\end{bmatrix} = \begin{bmatrix}-48\\ 37\\ 26\end{bmatrix} = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

Sol$^{n}$ (**)

$$\begin{bmatrix}-17 & -4 & -1\\ 14 & 3 & 1\\ 10 & 2 & 1\end{bmatrix}\begin{bmatrix}1\\ 3\\ 0\end{bmatrix} = \begin{bmatrix}-29\\ 23\\ 16\end{bmatrix} = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$$